Kripke Models of Models of Peano Arithmetic

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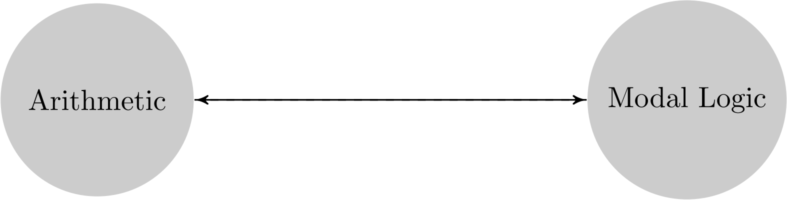
Provability logic



Provability predicate of PA

Peano Arithmetic (PA) — first–order theory of arithmetic.

There is a formula $\mathsf{Pr}(x)$ of $\mathcal{L}_{\mathsf{PA}}$ such that:

$$\vdash_{\mathsf{PA}} \varphi \;\Leftrightarrow\; \vdash_{\mathsf{PA}} \mathsf{Pr}(\overline{\ulcorner \varphi \urcorner}) \tag{1}$$

Furthermore,

(2) $\vdash_{\mathsf{PA}} \mathrm{Pr}(\overline{\ulcorner \varphi \to \psi \urcorner}) \wedge \mathrm{Pr}(\overline{\ulcorner \varphi \urcorner}) \to \mathrm{Pr}(\overline{\ulcorner \psi \urcorner})$

(3) $\vdash_{\mathsf{PA}} \mathrm{Pr}(\overline{\ulcorner \varphi \urcorner}) \to \mathrm{Pr}(\overline{\ulcorner \mathrm{Pr}(\overline{\ulcorner \varphi \urcorner}) \urcorner})$

Provability logic GL

The system GL (**G**ödel and **L**öb) is K plus Löb's axiom

$$\Box(\Box A \to A) \to \Box A.$$

GL is sound and (weakly) complete w.r.t. Kripke frames where R is transitive and conversely well–founded.

Arithmetical realisations

Definition

An *arithmetical realisation* is a function $* : \mathsf{prop}(\mathcal{L}_\Box) \to \mathsf{sent}(\mathcal{L}_{\mathsf{PA}})$. It is extended to a function from the full modal language by requiring:

i. $\bot^* = \bot$

ii. $(A \to B)^* = (A^* \to B^*)$

iii. $\Box A^* = \mathsf{Pr}\left(\ulcorner A^* \urcorner\right)$

Arithmetical soundness and completeness of GL

Theorem (Solovay)

GL *is the provability logic of* PA*:*

$$\vdash_{\mathsf{GL}} A \Leftrightarrow \text{ for all arithmetical realisations } *, \vdash_{\mathsf{PA}} A^*$$

Thus, in the context of PA,

- $\Box\varphi$ means: φ is provable ($\mathsf{Pr}(\overline{\ulcorner\varphi\urcorner})$).
- $\Diamond\varphi$ means: φ is consistent ($\neg\mathsf{Pr}(\overline{\ulcorner\neg\varphi\urcorner})$)

Goal of this talk

There is a relation $\rhd_{\mathsf{t}}$ (the t–internal model relation) between models of PA such that for all $\varphi \in \mathcal{L}_{\mathsf{PA}}$,

$$\mathcal{M} \models \mathsf{Pr}(\overline{\ulcorner \varphi \urcorner}) \Leftrightarrow \text{ for all } \mathcal{N} \text{ such that } \mathcal{M} \rhd_{\mathsf{t}} \mathcal{N},\ \mathcal{N} \models \varphi$$

$\Rrightarrow$ new interpretation of the modal operators in PA

- $\Box\varphi$ means: φ is true in all t–internal models
- $\Diamond\varphi$ means: φ is true in some t–internal model

The collection of models of PA, together with the relation $\rhd_{\mathsf{t}}$, can be seen as a big Kripke frame
$\Rrightarrow$ new perspective on Solovay's Theorem

t–INTERNAL MODELS

Relative translations

Definition

Let Σ, Θ be signatures. A *relative translation* $j\colon \Sigma \to \Theta$ is a tuple $\langle \delta, \tau \rangle$, where

1. δ is a Θ–formula with one free variable,
2. τ associates to each n-ary relation symbol R of Σ a formula R^τ of Θ with n free variables

τ is extended to a function from all formulas of Σ by requiring that it commutes with the propositional connectives, and furthermore

i. $(Rx_0 \ldots x_{n-1})^\tau = R^\tau(x_0 \ldots x_{n-1})$

ii. $(\forall x\, \varphi)^\tau = \forall x\, (\delta(x) \to \varphi^\tau)$

A relative translation $j\colon \Sigma \to \Theta$ is a way of uniformly defining a model $\mathcal{M}^j$ of signature Σ inside a given model $\mathcal{M}$ of signature Θ, provided that $\mathcal{M} \models \exists x\, \delta(x)$. We say that $\mathcal{M}^j$ is an *internal model* of $\mathcal{M}$.

t–internal models

Definition

$\mathcal{N}$ is a t–internal model of $\mathcal{M}$ ($\mathcal{M} \rhd_{\mathsf{t}} \mathcal{N}$) if there exists a triple $j = \langle \delta, \tau, \mathsf{tr} \rangle$ such that

1. $\langle \delta, \tau \rangle$ is a relative translation from the signature of $\mathcal{N}$ to the signature of $\mathcal{M}$, and $\mathcal{N} = \mathcal{M}^{\langle \delta, \tau \rangle}$.
2. tr is a formula of the signature of $\mathcal{M}$, and the following sentences are satisfied in $\mathcal{M}$:
   i. $\forall x \, (\delta(x) \to (R^{\tau} x \leftrightarrow \mathsf{tr}(Rc_x)))$
   ii. $\forall \varphi \in \mathsf{sent}, \forall \psi \in \mathsf{sent} \, (\mathsf{tr}(\varphi \to \psi) \leftrightarrow (\mathsf{tr}(\varphi) \to \mathsf{tr}(\psi)))$
   iii. $\forall \varphi \in \mathsf{sent} \, (\mathsf{tr}(\neg \varphi) \leftrightarrow \neg \mathsf{tr}(\varphi))$
   iv. $\forall \varphi \in \mathsf{sent}, \forall u \in \mathsf{var} \; (\mathsf{tr}(\forall u \, \varphi) \leftrightarrow \forall x \, (\delta(x) \to \mathsf{tr}(\varphi(c_x))))$
   v. $\forall \varphi \in \mathsf{sent} \, (\mathsf{Ax}_{\mathsf{PA}}(\varphi) \to \mathsf{tr}(\varphi))$

We write $j \colon \mathcal{M} \rhd_{\mathsf{t}} \mathcal{N}$, and refer to the components of j by δ_j, τ_j, tr_j.

An arithmetical accessibility relation

Theorem

Let $\mathcal{M} \models \mathsf{PA}$. *Then for any* $\mathcal{L}_{\mathsf{PA}}$*–sentence* φ,

$$\mathcal{M} \models \mathsf{Pr}(\varphi) \Leftrightarrow \textit{ for all } \mathcal{N} \textit{ with } \mathcal{M} \rhd_{\mathsf{t}} \mathcal{N},\ \mathcal{N} \models \varphi$$

An arithmetical accessibility relation (1)

Lemma

Let $j\colon \mathcal{M} \rhd_{\mathsf{t}} \mathcal{N}$. *For any* $\mathcal{L}_{\mathsf{PA}}$*–sentence* φ, $\mathcal{M} \models \mathsf{Pr}(\varphi) \to \varphi^{\tau_j}$

Theorem (1)

Let $\mathcal{M} \models \mathsf{PA}$. *Then for any* $\mathcal{L}_{\mathsf{PA}}$*–sentence* φ,

$$\mathcal{M} \models \mathsf{Pr}(\varphi) \Rightarrow \text{ for all } \mathcal{N} \text{ with } \mathcal{M} \rhd_{\mathsf{t}} \mathcal{N},\ \mathcal{N} \models \varphi$$

Proof.

Let $\mathcal{M} \models \mathsf{PA}$ and $\mathcal{M} \models \mathsf{Pr}(\varphi)$. Let j: $\mathcal{M} \rhd_{\mathsf{t}} \mathcal{N}$. By the Lemma, $\mathcal{M} \models \varphi^{\tau_j}$, whence $\mathcal{N} \models \varphi$ by the internal model construction. ❑

An arithmetical accessibility relation

Theorem (2)

Let $\mathcal{M} \models \mathsf{PA}$*. Then for any* $\mathcal{L}_{\mathsf{PA}}$*–sentence* φ*,*

$$\mathcal{M} \models \neg \mathsf{Pr}(\varphi) \Rightarrow \textit{ there is some } \mathcal{N} \textit{ with } \mathcal{M} \rhd_{\mathsf{t}} \mathcal{N}, \textit{ and } \mathcal{N} \models \neg\varphi$$

Proof.

If $\mathcal{M} \models \neg \mathsf{Pr}(\varphi)$, then $\mathcal{M} \models \mathsf{Con}(\mathsf{PA} + \neg\varphi)$. The proof is by the arithmetised Gödel's Completeness Theorem, noting that the formula representing the Henkin set can be viewed as a truth definition for the internally constructed $\mathcal{N} \models \mathsf{PA} + \neg\varphi$. ❑

The big model

Defining big Kripke models

Define the Kripke frame $\mathfrak{F}_{\mathsf{big}}$ as follows:

i. W_{big} consists of all models of PA

ii. R_{big} is $\rhd_{\mathsf{t}}$

Let $*$ be an arithmetical realisation. We turn $\mathfrak{F}_{\mathsf{big}}$ into a model $\mathfrak{M}^*_{\mathsf{big}}$ by letting:

$$\langle W_{\mathsf{big}}, \rhd_{\mathsf{t}} \rangle, \mathcal{M} \Vdash^* p :\Leftrightarrow \mathcal{M} \models p^*$$

By the properties of $\rhd_{\mathsf{t}}$, we get for any $A \in \mathcal{L}_{\Box}$:

$$\langle W_{\mathsf{big}}, \rhd_{\mathsf{t}} \rangle, \mathcal{M} \Vdash^* A \Leftrightarrow \mathcal{M} \models A^*.$$

Relation to Solovay's proof

For any GL–model M with domain $\{1,\dots,n\}$ and root 1, let M^0 denote the model M, where 0 is added as a root below 1.

Theorem

For any GL*–model* $M = \langle\{1,\dots,n\}, R, V\rangle$*, there exists an arithmetical realisation* $*$*, and a relation* $Z\colon\{0,1,\dots,n\}\times W_{\mathsf{big}}$ *such that* Z *is a total bisimulation between* M^0 *and* $\mathfrak{M}^*_{\mathsf{big}}$*.*

Proof.

Let $S_0,\cdots,S_n$ be the Solovay sentences corresponding to M^0 and let $*$ be the Solovay realisation, i.e. $p^* = \bigvee_{i\colon M^0,i\Vdash p} S_i$. Let

$$(i,\mathcal{M})\in Z :\Leftrightarrow \mathcal{M}\models S_i. \qquad \square$$

Corollary (Arithmetical Completeness of GL)

If $\nvdash_{\mathsf{GL}} A$*, then* $\nvdash_{\mathsf{PA}} A^*$ *for some arithmetical realisation* $*$*.*