Rules of Inference

Lecture 3

Thursday, September 26

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TbiLLC 2013
Gudauri, Georgia, September 23-27, 2013

Today

- Unification theory
- Proof systems
- Open problems

Unification theory

Unification theory

Given two terms s and t, is there a substitution σ such that $\sigma s =_{\mathsf{E}} \sigma t$?

Applications: computer science, linguistics, ...

In a logic L: $A =_E B$ is $\vdash_{\mathsf{L}} A \leftrightarrow B$.

The study of substitutions σ such that $\vdash_{\mathsf{L}} \sigma A$.

Dfn σ is a *unifier* of A iff $\vdash \sigma A$.

Dfn $\tau \leqslant \sigma$ iff for some τ' for all atoms p: $\vdash \tau(p) \leftrightarrow \tau'\sigma(p)$.

Dfn σ is a *maximal* unifier (mu) of A if among the unifiers of A it is maximal.

Dfn A unifier σ of A is a *mgu* if $\tau \leqslant \sigma$ for all unifiers τ of A.

Dfn A set of unifiers is *complete* for A if all unifiers of A are less general ($\leqslant$) than a unifier in the set.

Note Projective unifiers are most general unifiers.

Unification types

Ex p has a mgu in any intermediate logic: $\sigma(p) = \top$. In intermediate logics, every consistent formula has a unifier.

Dfn A logic has unification type

unitary if every unifiable formula has a mgu;

finitary if it is not unitary and every unifiable formula has a finite complete set of mus;

infinitary if it is not finitary and every unifiable formula has a finite or infinite complete set of mus;

nullary if none of the above.

Thm If $v_I(A) = 1$, then σ_I^A is a mgu of A in CPC.

Cor CPC has unitary unification.

No unitary unification

Note IPC does not have unitary unification.
Prf $A = p \vee \neg p$ has no mgu. For consider σ_0 and σ_1 where

$$\sigma_0(p) = \top \quad \sigma_0(q) = q \qquad \sigma_1(p) = p \quad \sigma_1(q) = \bot.$$

Neither $\sigma_0 \leqslant \sigma_1$ nor $\sigma_1 \leqslant \sigma_0$. So A has no mgu.
If τ is unifier of A, then because of the disjunction property $\vdash_{\mathsf{IPC}} \tau p$ or $\vdash_{\mathsf{IPC}} \neg\tau p$. Hence $\tau \leqslant \sigma_0$ or $\tau \leqslant \sigma_1$. Hence A has a finite set of mus: $\{\sigma_0, \sigma_1\}$.

Note Many modal logics do not have unitary unification.

Thm If there is a set of admissible rules $\mathcal{R}$ such that for every formula A there is a finite set of projective formulas Π_A such that

$$\bigvee \Pi_A \vdash_{\mathsf{L}} A \vdash_{\mathsf{L}}^{\mathcal{R}} \Pi_A,$$

then $\mathcal{R}$ is a basis for the admissible rules of L and L has finitary or unitary unification.

Prf
Let for $B \in \Pi_A$, σ_B be its projective unifier and let $\mathcal{C}$ be the set of these unifiers. If τ is a unifier of A, then it has to be a unifier of B for some $B \in \Pi_A$. Therefore $\tau \leqslant \sigma_B$, proving that $\mathcal{C}$ is complete. $\square$

Thm (Ghilardi)
The unification type of IPC, K4, S4, GL is finitary. In KC and S4.3 it is unitary.

Thm (Dzik '06)
All intermediate logics with unitary unification are extensions of KC. All intermediate logics with finitary unification are extensions of the logic of the fork. Similar for S4.3 in modal logic.

Thm (Marra & Spada '11)
Łukasiewicz logic has nullary unification type. Formulas do not have finite projective approximations.

Proof systems

Proof systems

In many intermediate and modal logics admissibility is decidable.

Many intermediate and modal logics have a decent basis for their admissible rules.

Can admissibility in these logics be captured by a decent proof system?

Yes: a sequent calculus to reason about rules consisting of sequents.

Dfn A *generalized sequent rule (gs-rule)* is an expression

$$\mathcal{G} \triangleright \mathcal{H},$$

where $\mathcal{G}$ and $\mathcal{H}$ are sets of sequents.

$\mathcal{G} \triangleright \mathcal{H}$ is *admissible* if $\mathcal{G} \mathrel{|\!\sim} \mathcal{H}$, which is short for

$$\{I(S) \mid S \in \mathcal{G}\} \mathrel{|\!\sim} \{I(S) \mid S \in \mathcal{H}\}.$$

Aim:
A proof system GAL for gs-rules such that $\vdash_{\mathsf{GAL}} \mathcal{G} \triangleright \mathcal{H}$ iff $\mathcal{G} \mathrel{|\!\sim}_{\mathsf{L}} \mathcal{H}$.

Dfn For a modal logic L, GAL consists of (G3 a calculus for CPC):

Right Logical Rules

$$\frac{\mathcal{G} \rhd S_1, \mathcal{H} \quad \mathcal{G} \rhd S_2, \mathcal{H}}{\mathcal{G} \rhd S, \mathcal{H}} \text{ for every rule } \frac{S_1 \quad S_2}{S} \text{ of G3}$$

Left Logical Rules

$$\frac{\mathcal{G}, S_1, S_2 \rhd \mathcal{H}}{\mathcal{G}, S \rhd \mathcal{H}} \text{ for every rule } \frac{S_1 \quad S_2}{S} \text{ of G3}$$

Visser Rules

$$\frac{\mathcal{G}, S, S_1 \rhd \mathcal{H} \quad \dots \quad \mathcal{G}, S, S_n \rhd \mathcal{H}}{\mathcal{G}, S \rhd \mathcal{H}} \text{ if } \frac{S}{\{S_1, \dots, S_n\}} \text{ is a rule in the basis}$$

and some other rules ...

Thm GAK4, GAS4 and GAGL are sound and complete proof systems for admissibility in K4, S4 and GL.

Cor Admissibility in K4, S4 and GL is decidable.

Open problems: to (wish to) do

Ways

Four approaches:

- characteristic
- categorical
- canonical
- syntactic

Dfn The language $\mathcal{L}$ consists of predicate and function symbols, variables, the connectives $\wedge, \vee, \rightarrow, \neg$ and the quantifiers $\exists, \forall$.

Possible requirements on substitutions:

A substitutions σ is a map from $\mathcal{F}_{\mathcal{L}}$ to $\mathcal{F}_{\mathcal{L}}$ that commutes with the connectives and quantifiers and such that ...

- $\sigma(P(t_1,\ldots,t_n)) = \sigma(P(x_1,\ldots,x_n))[t_1/x_1,\ldots,t_n/x_n]$;
- other requirements?

Thm The *Skolem rule* $\exists x A(x, fx)/\exists x \forall y A(x, y)$ (f fresh) is admissible in classical predicate logic.

Thm (Avigad '03)
If a theory can code finite functions, then the Skolem rule cannot shorten proofs more than polynomially.

Thm (Baaz & Hetzl & Weller '12)
In the setting of sequent calculi and cut-free proofs, the Skolem rule exponentially shortens proofs.

Thm (Iemhoff & Visser '01)
The Visser rules form a basis for the propositional admissible rules of Heyting Arithmetic.

Thm (Visser '99)
Admssibility in Heyting Arithmetic is Π_2-complete.

Ex An infinite admissible rule for Heyting Arithmetic:

$$\frac{\Gamma \Rightarrow \exists x A x}{\{\Gamma \Rightarrow At \mid t \text{ a term}\} \cup \{\Gamma \Rightarrow A \mid A \in \Gamma^a\}} \quad (\Gamma \text{ implications only})$$

Thm (Jeřábek '11) K has nullary unification.

$p \to \Box p$ is admissibly saturated but not projective (not exact).

It's a small world?

All these other logics?

Why

Provide meta-mathematical reasons for the admissibility of certain rules.

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