Herbrand's Theorem via Hypercanonical Extensions

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Outline

1. Introduction to substructural logics
2. Algebraic proof theory for substructural logics
3. Herbrand's theorem via hypercanonical extensions

Introduction to Substructural Logics

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- Abundance of weird logics/algebras
  - allows us to speak of criteria for various properties

- The base system for substructural logics (Ono 90) $\approx$ Intuitionistic logic without structural rules.
- Formulas:

$$\varphi, \psi \ ::= \ p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \cdot \psi \mid \varphi \backslash \psi \mid \varphi / \psi \mid 1 \mid 0$$

- 0 is used to define negations:

$$-a = a \backslash 0, \qquad \sim a = 0/a.$$

- Sequents: $\Gamma \Rightarrow \Pi$
  (Γ: sequence of formulas, Π: at most one formula)

Inference Rules of FL

$$\frac{\Gamma \Rightarrow A \quad \Delta_1, A, \Delta_2 \Rightarrow \Pi}{\Delta_1, \Gamma, \Delta_2 \Rightarrow \Pi}\ (Cut) \qquad \frac{}{A \Rightarrow A}\ (Id)$$

$$\frac{\Gamma_1, A, \Gamma_2 \Rightarrow \Pi \quad \Gamma_1, B, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, A \vee B, \Gamma_2 \Rightarrow \Pi}\ (\vee l) \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ (\vee r)$$

$$\frac{\Gamma_1, A_i, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, A_1 \wedge A_2, \Gamma_2 \Rightarrow \Pi}\ (\wedge l) \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ (\wedge r)$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, A \cdot B, \Gamma_2 \Rightarrow \Pi}\ (\cdot l) \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \cdot B}\ (\cdot r)$$

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$$\frac{\Gamma \Rightarrow A \quad \Delta_1, B, \Delta_2 \Rightarrow \Pi}{\Delta_1, B/A, \Gamma, \Delta_2 \Rightarrow \Pi}\ (/l) \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow B/A}\ (/r)$$

$$\frac{\Gamma \Rightarrow \Pi}{\Gamma_1, 1, \Gamma_2 \Rightarrow \Pi}\ (1l) \qquad \frac{}{\Rightarrow 1}\ (1r) \qquad \frac{}{0 \Rightarrow}\ (0l) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (0r)$$

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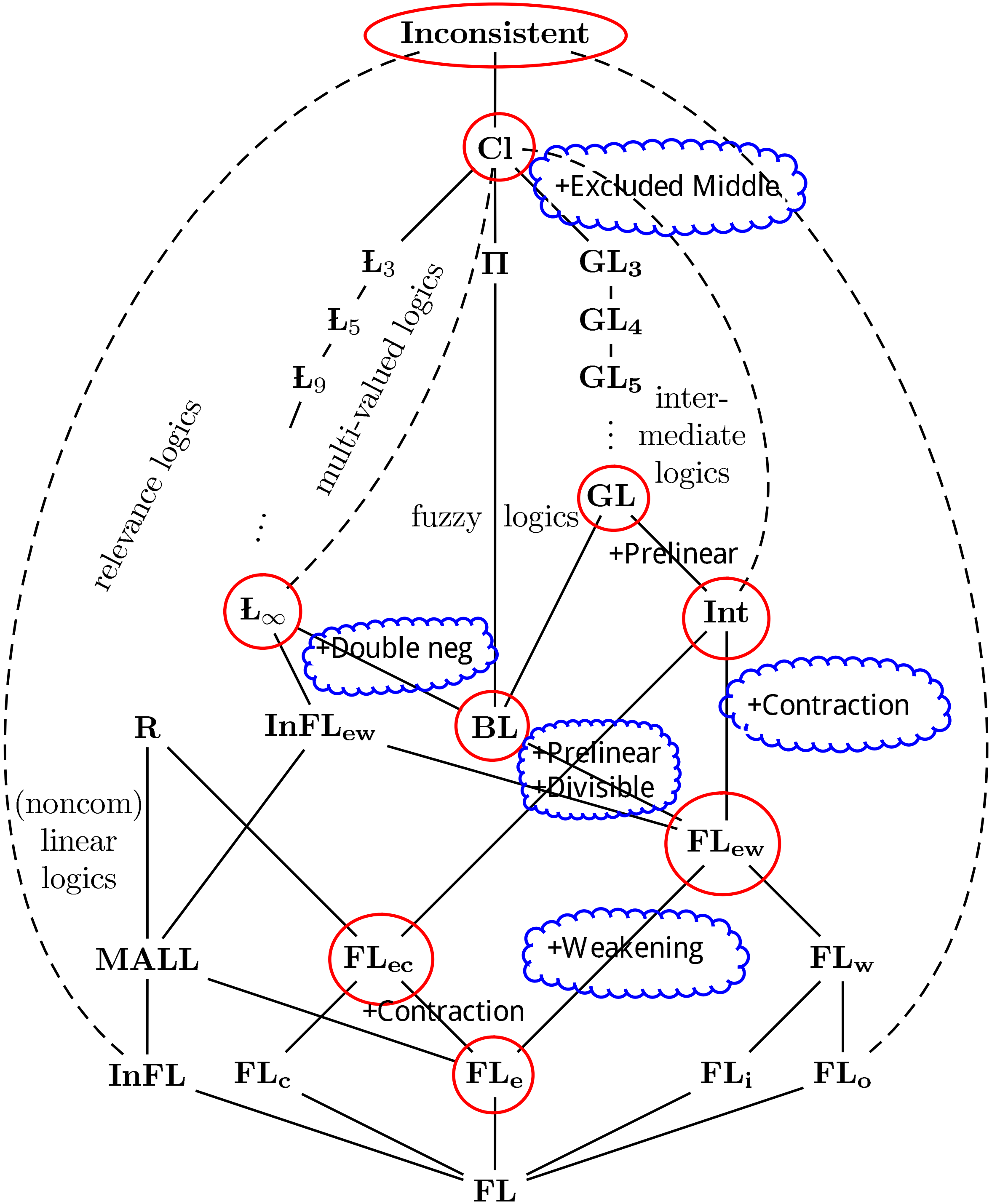
Some axioms:

(e)	$\varphi \cdot \psi \to \psi \cdot \varphi$	(exchange)
(w)	$\varphi \to 1, \qquad 0 \to \varphi$	(weakening)
(c)	$\varphi \to \varphi \cdot \varphi$	(contraction)
(in)	$\neg\neg\varphi \to \varphi$	(involutivity)
(dist)	$\varphi \wedge (\psi_1 \vee \psi_2) \to (\varphi \wedge \psi_1) \vee (\varphi \vee \psi_2)$	(distributivity)
(pl)	$(\varphi \to \psi) \vee (\psi \to \varphi)$	(prelinearity)
(div)	$\varphi \wedge \psi \to \varphi \cdot (\varphi \to \psi)$	(divisibility)



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Residuated Lattices:
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Residuated Lattices

Residuated lattice: $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ such that

- $\langle A, \wedge, \vee\rangle$ is a lattice;
- $\langle A, \cdot, 1\rangle$ is a monoid;
- $a \cdot b \leq c \;\Leftrightarrow\; b \leq a\backslash c \;\Leftrightarrow a \leq c/b$.

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Some identities:

- $a \cdot (a\backslash b) \leq b$
- $(a \vee b) \cdot c = (a \cdot c) \vee (b \cdot c)$
- $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$
- $(a \vee b) \to c = (a \to c) \wedge (b \to c)$

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- $(a \vee b) \to c = (a \to c) \wedge (b \to c)$
- $a\backslash b = b/a$ (with (e))
- $a \cdot b \leq a \wedge b$ (with (w))
- $a \cdot b \geq a \wedge b$ (with (c))

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- H: homomorphic images
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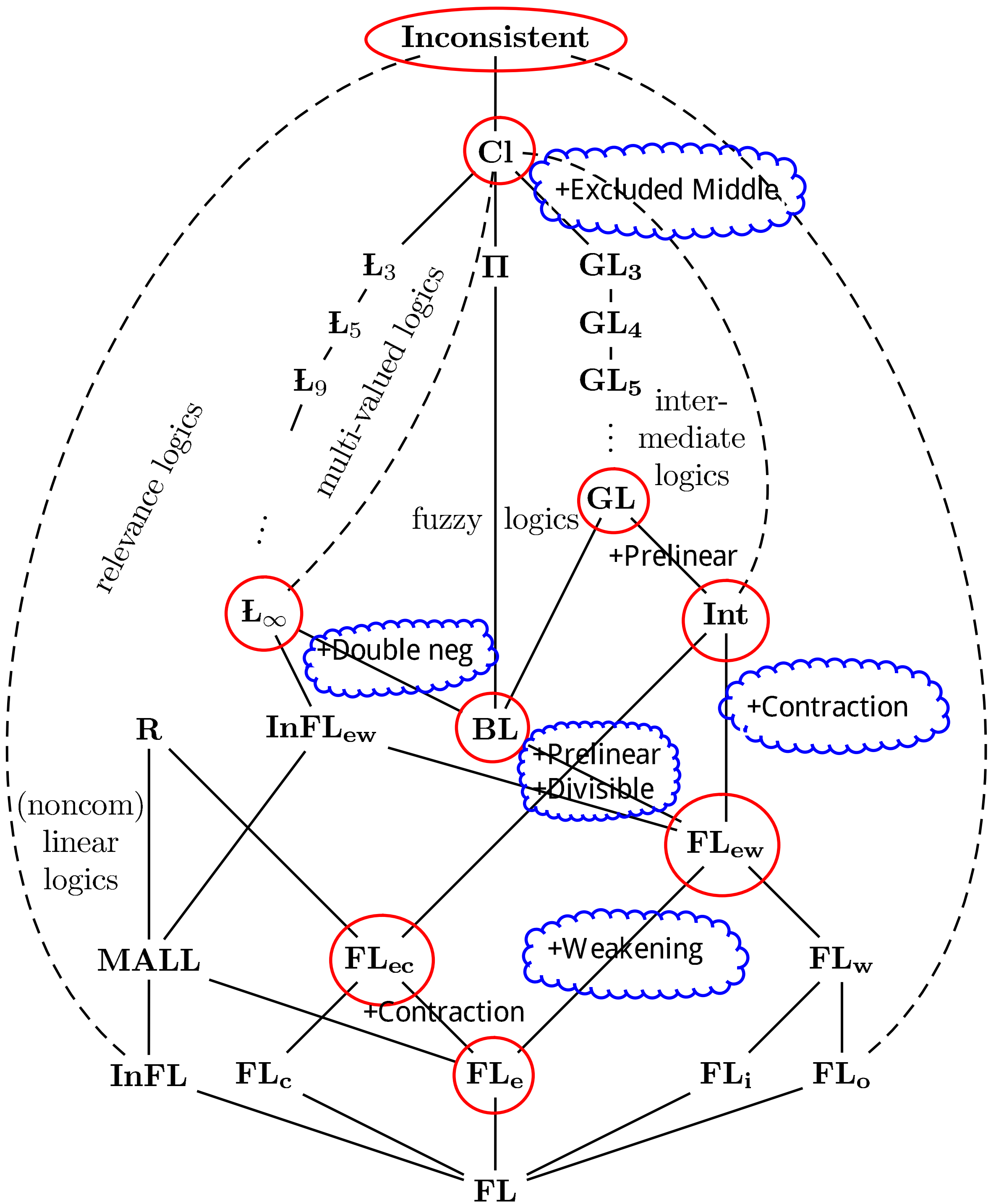
FL := the variety of FL algebras.

Algebraization Theorem

The substructural logics are in 1-1 correspondence with the subvarieties of FL. If $\mathbf{L}$ corresponds to V,

$$\Phi \vdash_{\mathbf{L}} \psi \quad \text{iff} \quad 1 \leq \Phi \models_{\mathsf{V}} 1 \leq \psi.$$



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Theorem (Horcik-T. 11)

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Dichotomy Problem

Is there a substructural logic which is neither coNP-complete nor PSPACE-hard?

Algebraic Proof Theory for Substructural Logics

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Our aim: to salvage those brilliant ideas from the sea of bureaucracy.

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Algebraic proof theory for SL

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- Today I will focus on cut elimination ≈ algebraic completion.

Completions

Let $\mathbf{A}$ be an FL algebra.
A completion of $\mathbf{A}$ is a pair of a complete FL algebra $\mathbf{B}$ and an embedding $e : \mathbf{A} \hookrightarrow \mathbf{B}$.

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We consider 4 types of completion:

- MacNeille completions
  (Dedekind, MacNeille, Schmidt, Banaschewski . . .)
- Canonical extensions
  (Tarski, Jónson, Gehrke, Harding . . .)
- Hyper-MacNeille completions
- Hypercanonical extensions

MacNeille completions

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Let $\mathbf{A}$ be a lattice. Its completion $\mathbf{B}$ is

- join-dense if for every $x \in B$, $x = \bigvee\{a \in A : a \leq x\}$.
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Theorem (Schmidt 56, Banaschewski 56)

Every lattice $\mathbf{A}$ has a join-dense and meet-dense completion $\overline{\mathbf{A}}$, unique up to isomorphism, called the MacNeille completion.
It can be extended to FL algebras too.

A preframe is $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$ such that

- $N \subseteq W \times W'$,
- $(W, \circ, \varepsilon)$ is a monoid, $\epsilon \in W'$.

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A residuated frame is a preframe where for every $x \in W, z \in W'$ there are elements $x \backslash\backslash z$ and $z /\!/ x \in W'$ such that

$$x \circ y \; N \; z \iff x \; N \; z /\!/ y \iff y \; N \; x \backslash\backslash z.$$

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Lemma

If $\mathbf{W}$ is a preframe, then

$$\begin{aligned} \tilde{\mathbf{W}} &:= (W,\ W \times W' \times W,\ \tilde{N}, \circ, \varepsilon, (\varepsilon, \epsilon, \varepsilon)) \\ x \; \tilde{N} \; (u, z, v) &\iff u \circ x \circ v \; N \; z \end{aligned}$$

is a residuated frame.

Residuated frames

Residuated frames are effective tools to build complete FL algeblras.

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Given $X \subseteq W$ and $Z \subseteq W'$,

$$\begin{array}{rcl} X^{\triangleright} & := & \{z \in W' : x \; N \; z \text{ for every } x \in X\} \\ Z^{\triangleleft} & := & \{x \in W : x \; N \; z \text{ for every } z \in Z\} \end{array}$$

$(^{\triangleright}, {}^{\triangleleft})$ forms a Galois connection:

$$X \subseteq Z^{\triangleleft} \iff X^{\triangleright} \supseteq Z$$

that induces a closure operator $\gamma(X) := X^{\triangleright\triangleleft}$ on $\wp(W)$.

Given $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$ and $X, Y \subseteq W$,

$G(W) :=$ the set of Galois-closed subsets of W
$(X = \gamma(X) = X^{\triangleright\triangleleft})$

$$\begin{array}{rcl} X\backslash Y & := & \{y : x \circ y \in Y \text{ for every } x \in X\} \\ Y/X & := & \{y : y \circ x \in Y \text{ for every } x \in X\} \\ X \circ_\gamma Y & := & \gamma(X \circ Y) \\ X \cup_\gamma Y & := & \gamma(X \cup Y) \end{array}$$

Lemma

$$\mathbf{W}^+ := (G(W), \cap, \cup_\gamma, \circ_\gamma, \backslash, /, \gamma(\varepsilon), \epsilon^{\triangleleft})$$

is a complete FL algebra, called the complex algebra of $\mathbf{W}$.

$Fm :=$ the set of formulas.
$Fm^* :=$ the set of formula sequences.

- Let $\mathbf{W}_{cf} := (Fm^*, Fm \cup \{\emptyset\}, N_{cf}, \circ, \emptyset, \emptyset)$ where

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Then $\mathbf{W}_{cf}^{+}$ is an FL algebra such that

$$\models_{\mathbf{W}_{cf}^{+}} 1 \leq \varphi \quad \text{implies} \quad \Rightarrow \varphi \text{ is cut-free derivable.}$$

$\Rightarrow$ Algebraic cut elimination.

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  ⇒ Algebraic cut elimination.

- Given an FL algebra $\mathbf{A}$, let $\mathbf{W}_{\mathbf{A}} := (A, A, \leq_{\mathbf{A}}, \cdot, 1, 0)$.

  Then $\mathbf{W}_{\mathbf{A}}^{+}$ is the MacNeille completion of $\mathbf{A}$.

From axioms to rules

Let $\mathbf{L}$ be a SL (axiomatic extension of $\mathbf{FL}$). To obtain an analytic calculus for $\mathbf{L}$, axioms have to be transformed into structural rules:

$$\varphi \to \varphi \cdot \varphi \qquad \varphi \cdot \varphi \to \varphi \qquad \neg(\varphi \wedge \neg\varphi)$$

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Let V be a subvariety of FL. To show that V is closed under completions, identities have to be transformed into quasi-identities:

$$x \le xx \qquad\qquad xx \le x \qquad\qquad x \wedge \neg x \le 0$$

$$\frac{xx \le z}{x \le z} \qquad \frac{x \le z \quad y \le z}{xy \le z} \qquad \frac{xx \le 0}{x \le 0}$$

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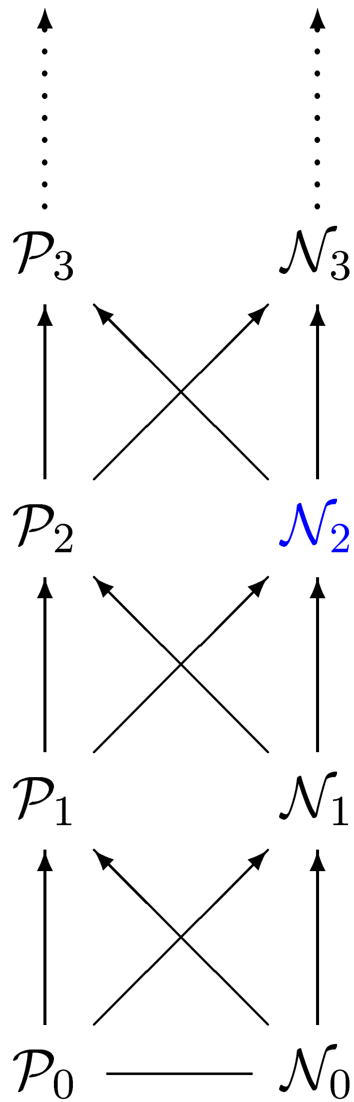
Fundamental Question

Which axioms/identities can be transformed into "good" structural rules/quasi-identities?

Substructural hierarchy



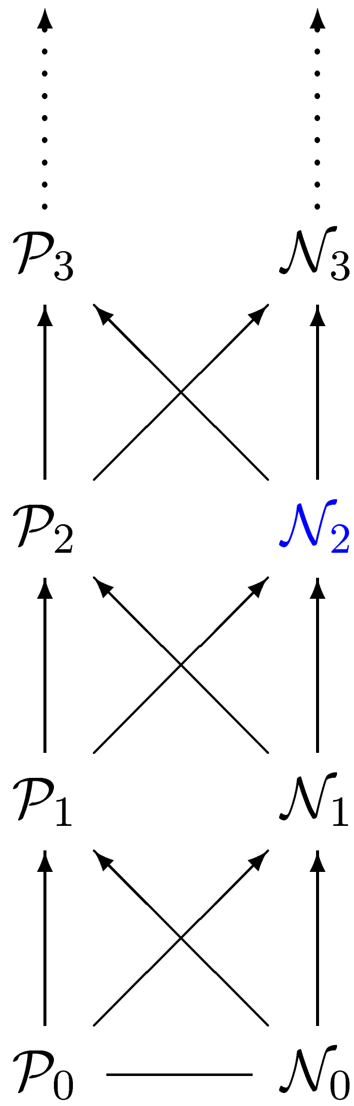
Classification of axioms

$$\begin{array}{rcl} \mathcal{P}_0, \mathcal{N}_0 & ::= & \text{the set of variables} \\ \mathcal{P}_n & ::= & \mathcal{N}_{n-1} \mid 1 \mid \mathcal{P}_n \vee \mathcal{P}_n \mid \mathcal{P}_n \cdot \mathcal{P}_n \\ \mathcal{N}_n & ::= & \mathcal{P}_{n-1} \mid 0 \mid \mathcal{N}_n \wedge \mathcal{N}_n \mid \mathcal{P}_n \to \mathcal{N}_n \end{array}$$

Some $\mathcal{N}_2$ axioms:

$\alpha \to 1$, $0 \to \alpha$	weakening
$\alpha \to \alpha \cdot \alpha$	contraction
$\alpha \cdot \alpha \to \alpha$	expansion
$\alpha^n \to \alpha^m$	knotted axioms $(n, m \geq 0)$
$\neg(\alpha \wedge \neg\alpha)$	no-contradiction

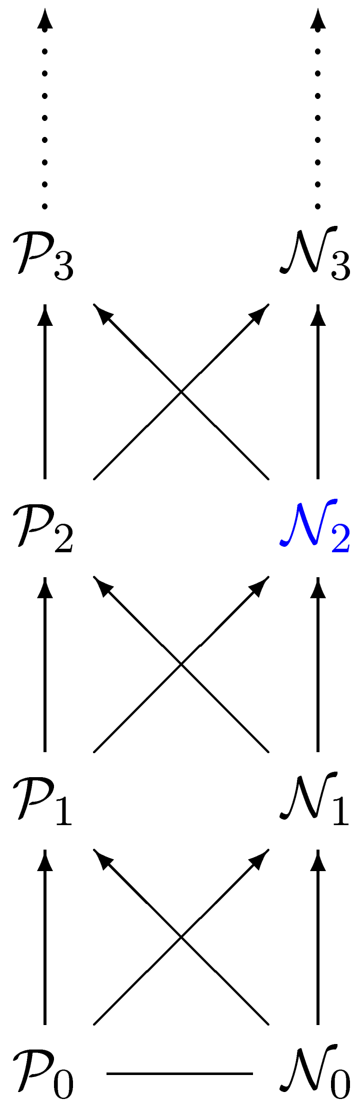
$\mathcal{N}_2$ corresponds to sequent calculus and MacNeille completions



Theorem (Ciabattoni-Galatos-T. 12)

1. Every $\mathcal{N}_2$ axiom can be transformed into a set of structural rules in sequent calculus $\mathbf{FL}$.

$\mathcal{N}_2$ corresponds to sequent calculus and MacNeille completions

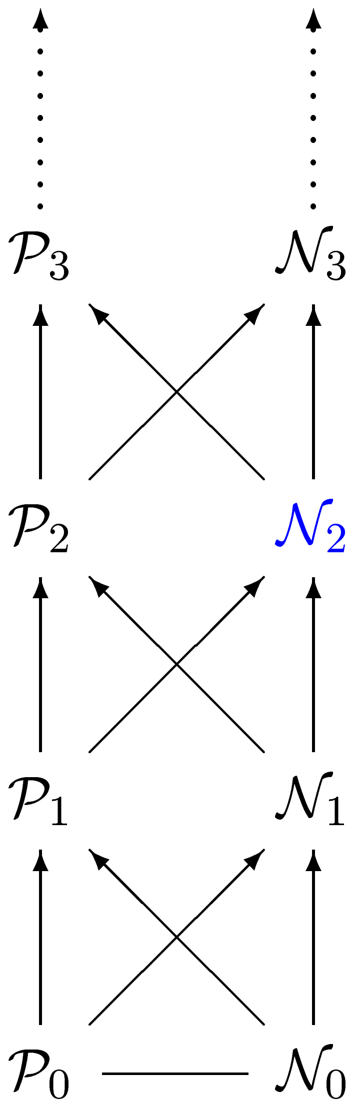


Theorem (Ciabattoni-Galatos-T. 12)

1. Every $\mathcal{N}_2$ axiom can be transformed into a set of structural rules in sequent calculus $\mathbf{FL}$.

2. For every set E of $\mathcal{N}_2$ axioms, the following are equivalent.

- □ $\mathbf{FL}(E)$ admits a strongly analytic sequent calculus (cut elimination for derivations with assumptions + subformula property).
- □ FL(E) is closed under MacNeille completions.
- □ E is acyclic (a syntactic criterion).



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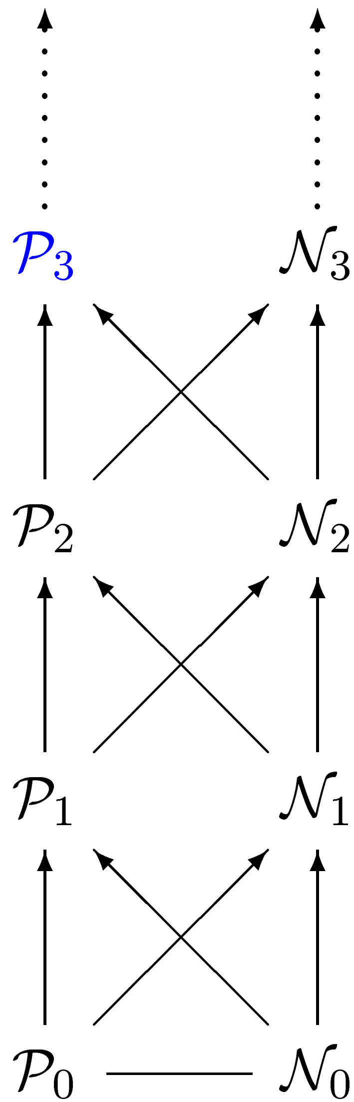
3. The above three hold whenever $(w) \in E$.

Let's climb up the hierarchy



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\end{array}
$$

Some $\mathcal{P}_3$ axioms:

$(\alpha \to \beta) \vee (\beta \to \alpha)$	prelinearity
$\alpha \vee \neg\alpha$	excluded middle
$\neg\alpha \vee \neg\neg\alpha$	weak excluded middle
$\neg(\alpha \cdot \beta) \vee (\alpha \wedge \beta \to \alpha \cdot \beta)$	weak nilpotent minimum
$\bigvee_{i=0}^{k}(\alpha_i \to \bigvee_{j \neq i} \alpha_j)$	bounded width $\leq k$
$\bigvee_{i=0}^{k}(\alpha_0 \wedge \cdots \wedge \alpha_{i-1} \to \alpha_i)$	bounded size $\leq k$

Limitation of sequent calculus/MacNeille completions

Fact

Every structural rule in sequent calculus is either derivable or contradictory in $\mathbf{Int}$.

$$\frac{\Gamma \Rightarrow \Pi \quad \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi} \qquad \frac{\Gamma \Rightarrow \Pi}{\Rightarrow \Pi}$$

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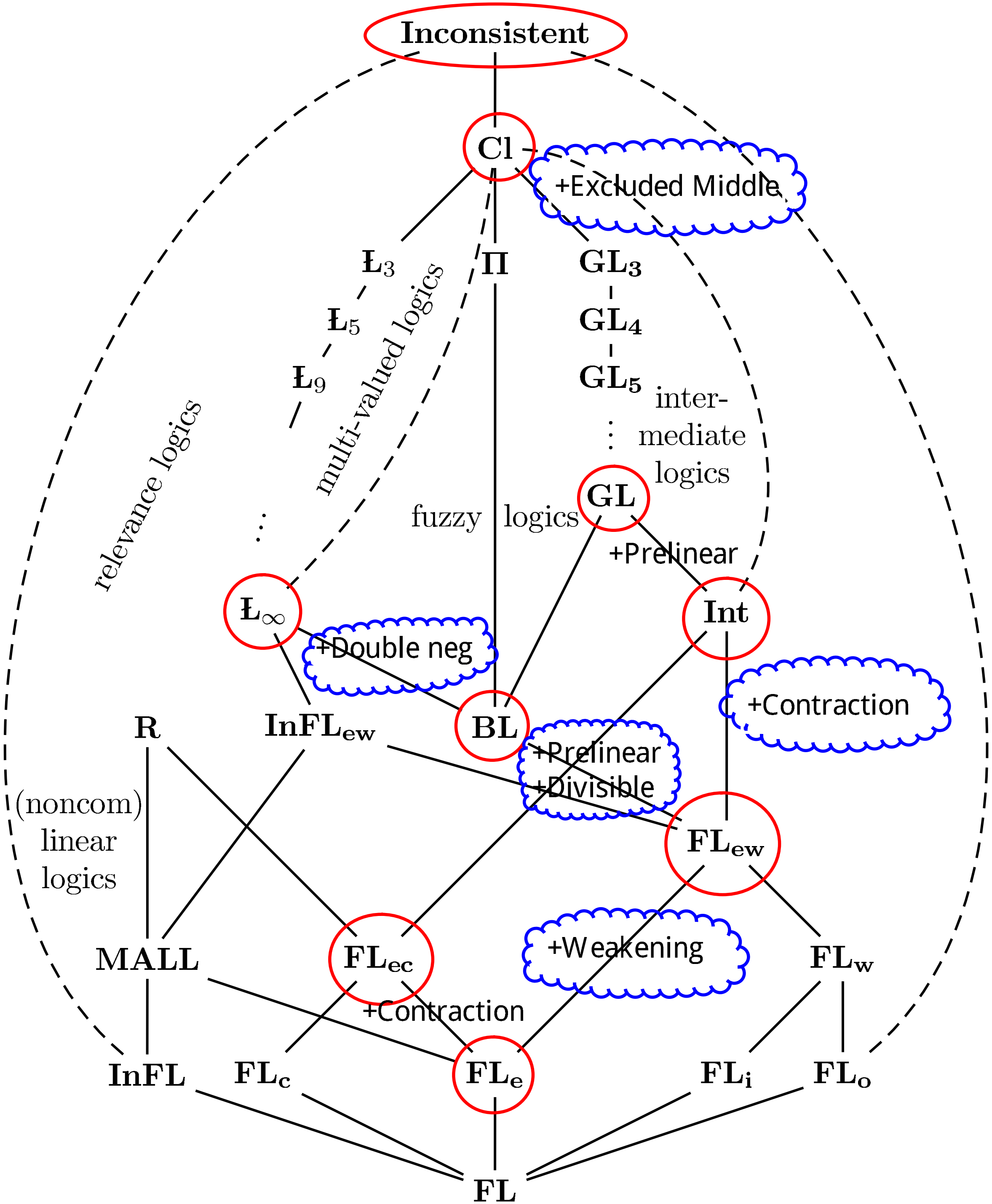
Theorem (G.Bezhanishvili-Harding 04)

There is no intermediate variety between HA and BA that is closed under MacNeille completions.

Eg. prelinearity cannot be dealt with by sequent calculus/MacNeille completions.



From
N. Galatos, P. Jipsen,
T. Kowalski and H. Ono,

Residuated Lattices:
An Algebraic Glimpse
at Substructural Logics, 2007.

- Hypersequents: $\Gamma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Gamma_m \Rightarrow \Pi_m$ (meaning $(\Gamma_1 \to \Pi_1) \vee \cdots \vee (\Gamma_m \to \Pi_m)$)

Hypersequent calculus (Avron 91)

- Hypersequents: $\Gamma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Gamma_m \Rightarrow \Pi_m$ (meaning $(\Gamma_1 \to \Pi_1) \vee \cdots \vee (\Gamma_m \to \Pi_m)$)
- Hypersequent calculus for $\mathbf{FL}$ consists of

Rules of $\mathbf{FL}$	Ext-Weakening	Ext-Contraction
$\dfrac{\Xi \mid \alpha, \Gamma \Rightarrow \beta}{\Xi \mid \Gamma \Rightarrow \alpha \to \beta}$	$\dfrac{\Xi}{\Xi \mid \Gamma \Rightarrow \Pi}$	$\dfrac{\Xi \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{\Xi \mid \Gamma \Rightarrow \Pi}$

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$$\frac{\Xi \mid \Gamma_1, \Delta_1 \Rightarrow \Pi \quad \Xi \mid \Gamma_2, \Delta_2 \Rightarrow \Lambda}{\Xi \mid \Gamma_1, \Gamma_2 \Rightarrow \Pi \mid \Delta_1, \Delta_2 \Rightarrow \Lambda}\ (com)$$

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is equivalent to

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- Cf. Ackermann Lemma-based Algorithm (Conradie-Palmigiano)
- Implemented by (Ciabattoni-Spendier)

Hyper-MacNeille completions

Let $\mathbf{A}$ be an FLew algebra. Define

$$\begin{aligned} \mathbf{W}^h_{\mathbf{A}} &:= (A \times A, A \times A, N, (\cdot, \vee), (1, 0), (0, 0)) \\ (a, h)\ N\ (b, k) &\iff 1 = (a \to b) \vee h \vee k \end{aligned}$$

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Theorem

$\mathbf{W}^{h+}_{\mathbf{A}}$ is a completion of $\mathbf{A}$, called the hyper-MacNeill completion.

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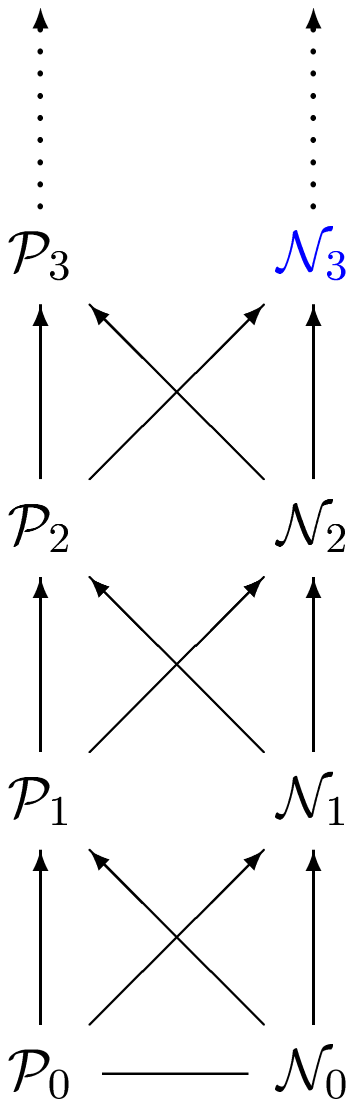
Theorem (CGT)

For every set E of $\mathcal{P}_3$ axioms,

- $\mathbf{FLew}(E)$ admits a strongly analytic hypersequent calculus.
- FLew(E) is closed under hyper-MacNeille completions.

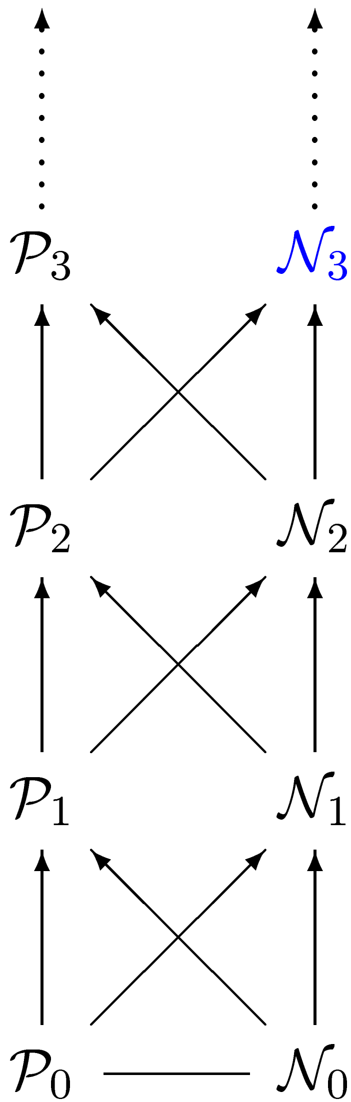
Class $\mathcal{N}_3$



Some $\mathcal{N}_3$ axioms:

$\alpha \wedge (\beta \vee \gamma) \to (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$	distributivity
$(\alpha \to \alpha \cdot \beta) \to \beta$	cancellativity
$\alpha \wedge \beta \to \alpha \cdot (\alpha \to \beta)$	divisibility

$$\mathbf{BL} := \mathbf{FLew} + (pl) + (div)$$
$$\text{Ł} := \mathbf{BL} + (in)$$



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Theorem (cf. Kowalski-Litak 08)

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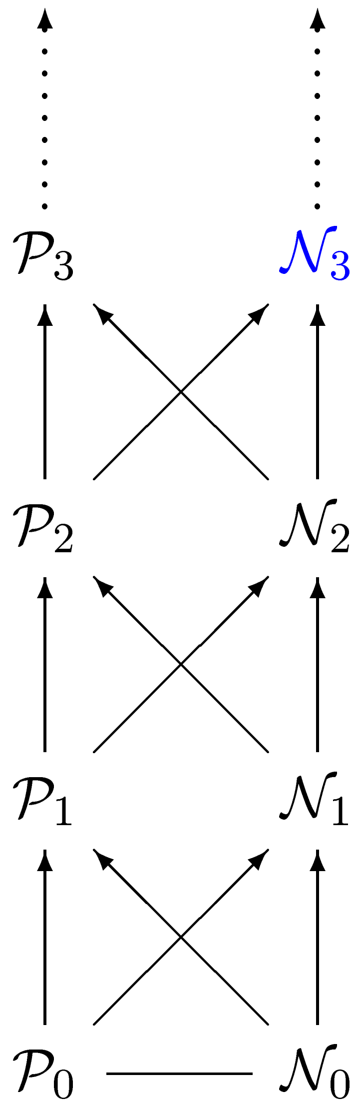
Class $\mathcal{N}_3$



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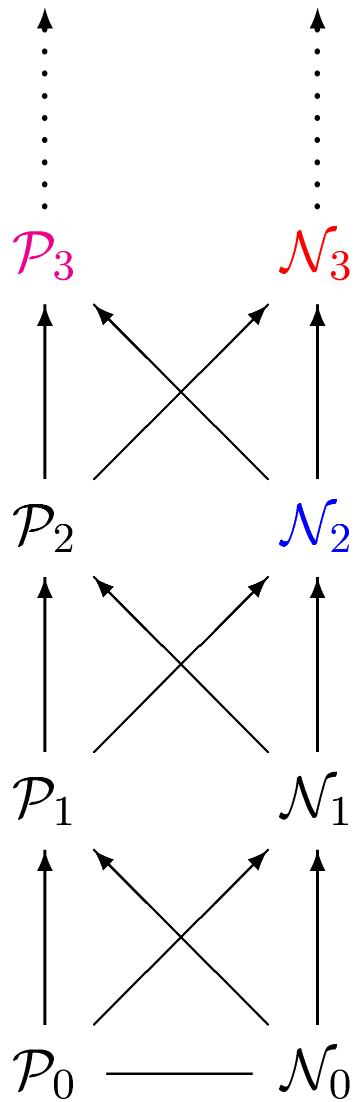
Limitation of uniform proof theory!

Summary

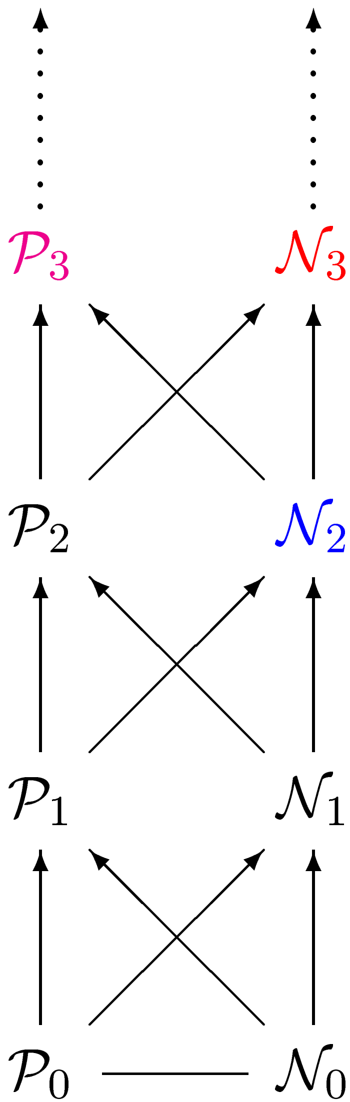


$\mathcal{N}_2$: sequent calculus
MacNeille completions

$\mathcal{P}_3$: hypersequent calculus
hyper-MacNeille completions

$\mathcal{N}_3$: limitation of uniform proof theory

Summary



$\mathcal{N}_2$: sequent calculus
MacNeille completions

$\mathcal{P}_3$: hypersequent calculus
hyper-MacNeille completions

$\mathcal{N}_3$: limitation of uniform proof theory

- Axioms $\Rightarrow$ rules is important in both proof theory and algebra.
- The hyper-construction (proof theory) is useful for completions (algebra) too.
- Limitation of proof theory is imposed by algebraic facts.

Herbrand's theorem via hypercanonical extensions

Let $\mathbf{L}$ be a propositional substructural logic.
$\mathbf{QL}$:= the predicate extension of $\mathbf{L}$ obtained by adding

$$\forall x.\alpha(x) \to \alpha(t) \qquad \alpha(t) \to \exists x.\alpha(x)$$

$$\frac{\beta \to \alpha(x)}{\beta \to \forall x.\alpha(x)} \qquad \frac{\alpha(x) \to \beta}{\exists x.\alpha(x) \to \beta} \qquad (x \text{ not free in } \beta)$$

Herbrand property

Definition

$\mathbf{L}$ satisfies the Herbrand property if for every set Ψ of universal formulas and every quantifier-free formula $\varphi(x)$,

$$\Psi \vdash_{\mathbf{QL}} \exists x.\varphi(x) \iff \Psi^\circ \vdash_{\mathbf{L}} \varphi(t_1) \vee \cdots \vee \varphi(t_n)$$

for some $t_1, \ldots, t_n$,

where $\Psi^\circ := \{\psi(\bar{t}) : \forall\overline{x}.\psi(\overline{x}) \in \Psi\}$.

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Herbrand property is related to compactness phenomena (previous talk).
What is the algebraic form of compactness?

Canonical extensions

Let $\mathbf{C}$ be a Boolean algebra and $X_{\mathbf{C}}$ be its Stone space. Then

$$\mathbf{C}^\sigma \quad := \quad (\mathcal{P}(X_{\mathbf{C}}), \cap, \cup, {}^C)$$

is a completion of $\mathbf{C}$.

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Let $\mathbf{D}$ be a bounded distributive lattice and $Y_{\mathbf{D}}$ be its Priestly space. Then

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$$\mathbf{D}^\sigma \quad := \quad (\mathcal{P}_{\downarrow}(Y_{\mathbf{D}}), \cap, \cup)$$

is a completion of $\mathbf{D}$.
Recall that

MacNeille completions $=$ join-dense, meet-dense completions

Do we have a similar abstract characterization for $\mathbf{C}^\sigma, \mathbf{D}^\sigma$?

Let $\mathbf{A}$ be a lattice. Its completion $\mathbf{B}$ is

- **dense** if for every $x \in B$, there exist $C_i, D_j \subseteq A$ $(i \in I, j \in J)$ such that

$$x = \bigvee_{i \in I} \bigwedge C_i = \bigwedge_{j \in J} \bigvee D_j.$$

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- compact if for every $C, D \subseteq A$,

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Theorem (Gehrke-Harding 01)

Every lattice $\mathbf{A}$ has a dense and compact completion $\mathbf{A}^\sigma$, unique up to isomorphism, called the canonical extension.

Herbrand property via compact completions

A completion of $\mathbf{A}$ is compact if for every $C, D \subseteq A$,

$$\bigwedge C \leq \bigvee D \Longrightarrow \bigwedge C_0 \leq \bigvee D_0$$

for some finite $C_0 \subseteq C$ and $D_0 \subseteq D$.

Theorem

If $\mathsf{V}(\mathbf{L})$ is closed under compact completions, then $\mathbf{L}$ satisfies the Herbrand property.

Theorem (Gehrke-Harding 01)

Let V be a variety of monotone lattice expansions.
If V is generated by a finite algebra, then V is closed under canonical extensions.

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Every finite-valued substructural logic satisfies the Herbrand property.

It actually applies to a much wider range of finite-valued logics.

Herbrand property for finite-valued logics

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The GH theorem is an algebraic counterpart of the uniform midsequent theorem for finite-valued logics (Baaz-Fermüller-Zach 94).

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Let V be a variety of bounded monotone lattice expansions. If V is closed under MacNeille completions, it is also closed under canonical extensions.

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Canonical extensions preserve (dist) $(\alpha \vee \beta) \wedge \gamma \leftrightarrow (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$.

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Corollary

Every substructural logic axiomatized by

- acyclic $\mathcal{N}_2$ axioms
- and/or (in), (dist)

satisfies the Herbrand property.

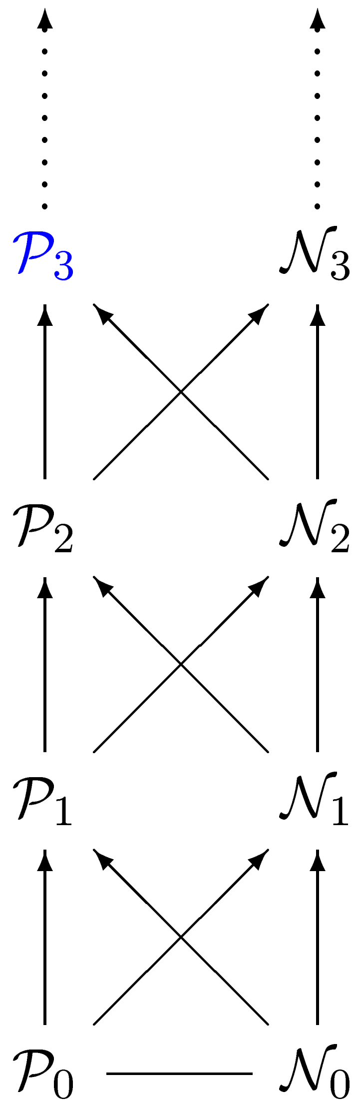
The GHV theorem states:

MacNeille completions $\Longrightarrow$ Canonical extensions.

It conforms to the proof theoretic intuition:

Cut elimination $\Longrightarrow$ Herbrand's theorem.

What about $\mathcal{P}_3$ logics?



Some $\mathcal{P}_3$ axioms:

$(\alpha \to \beta) \vee (\beta \to \alpha)$	prelinearity
$\alpha \vee \neg\alpha$	excluded middle
$\neg\alpha \vee \neg\neg\alpha$	weak excluded middle
$\neg(\alpha \cdot \beta) \vee (\alpha \wedge \beta \to \alpha \cdot \beta)$	weak nilpotent minimum
$\bigvee_{i=0}^{k}(\alpha_i \to \bigvee_{j\neq i} \alpha_j)$	bounded width $\leq k$
$\bigvee_{i=0}^{k}(\alpha_0 \wedge \cdots \wedge \alpha_{i-1} \to \alpha_i)$	bounded size $\leq k$

We want compact completions that preserve $\mathcal{P}_3$ axioms.

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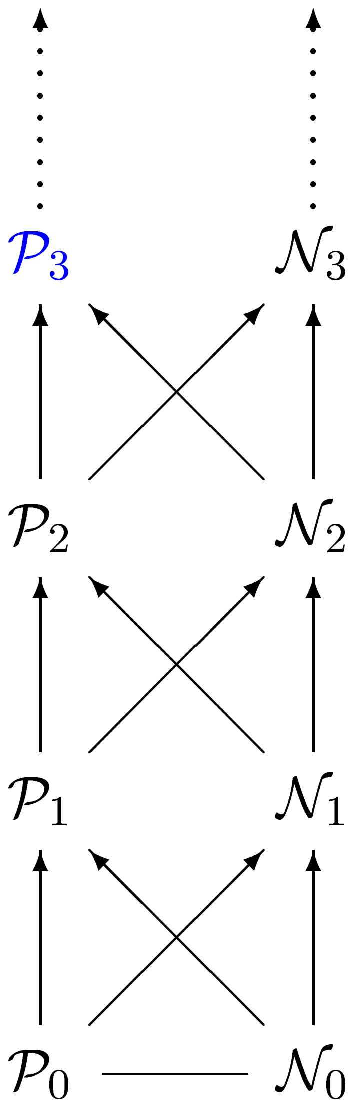

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We want compact completions that preserve $\mathcal{P}_3$ axioms.

$\Rightarrow$ Hypercanonical extensions.

Reminder: MacNeille and hyper-MacNeille completions

Let $\mathbf{A}$ be an FL algebra.

MacNeille completion of $\mathbf{A}$ is $\mathbf{W}^{+}_{\mathbf{A}}$ where

$$\mathbf{W_A} := (A, A, \leq, \cdot, 1, 0).$$

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$$\mathbf{W}_{\mathbf{A}} := (A, A, \leq, \cdot, 1, 0).$$

Assuming $\mathbf{A}$ is FLew, hyper-MacNeille completion is $\mathbf{W}_{\mathbf{A}}^{h+}$ where

$$\begin{array}{rcl} \mathbf{W}_{\mathbf{A}}^{h} & := & (A \times A, A \times A, N, \cdots) \\ (a, h)\ N\ (b, k) & \Longleftrightarrow & 1 = (a \to b) \vee h \vee k. \end{array}$$

Canonical extension of $\mathbf{A}$ is $\mathbf{W}_{\mathbf{A}}^{\sigma+}$ where

$$\begin{array}{rcl} \mathbf{W}_{\mathbf{A}}^{\sigma} & := & (\mathcal{F}_{\mathbf{A}}, \mathcal{I}_{\mathbf{A}}, N, \circ, \uparrow 1, \downarrow 0), \\ \mathcal{F}_{\mathbf{A}} & := & \text{the filters of } \mathbf{A} \\ \mathcal{I}_{\mathbf{A}} & := & \text{the ideals of } \mathbf{A} \\ f\ N\ i & \Longleftrightarrow & f \cap i \neq \emptyset \end{array}$$

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Assuming $\mathbf{A}$ is FLew, hypercanonical extension of $\mathbf{A}$ is $\mathbf{W}_{\mathbf{A}}^{H+}$ where

$$\begin{aligned} \mathbf{W}_{\mathbf{A}}^{H} &:= (\mathcal{F}_{\mathbf{A}} \times \mathcal{I}_{\mathbf{A}}, \mathcal{I}_{\mathbf{A}} \times \mathcal{I}_{\mathbf{A}}, N, \cdots) \\ (f, j)\ N\ (i, k) &\iff 1 \in (f \to i) \vee j \vee k \end{aligned}$$

Theorem

Hypercanonical extensions are compact completions. They preserve all $\mathcal{P}_3$ identities.

Herbrand property for $\mathcal{P}_3$ logics

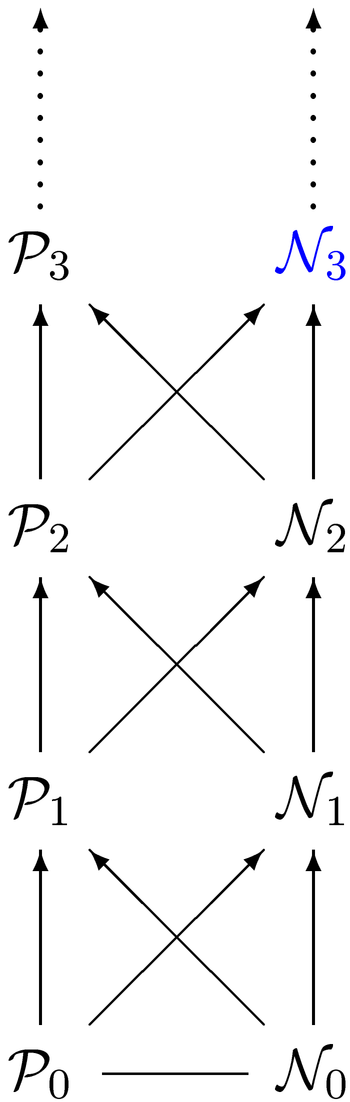
Theorem

Hypercanonical extensions are compact completions. They preserve all $\mathcal{P}_3$ identities.

Corollary

Every substructural logic over $\mathbf{FLew}$ axiomatized by $\mathcal{P}_3$ axioms satisfies the Herbrand property.

It applies to $\mathbf{MTL}$, $\mathbf{G}$, $\mathbf{LQ}$ and many more uniformly.



Recall that MV$(=$ V(Ł)$)$ is not closed under any completions.

Theorem (Baaz-Metcalfe 08)

Ł does not satisfy the Herbrand property, although it does satisfy an "approximate" Herbrand theorem.

Herbrand's theorem for ∃∀-formulas:

$$\Phi \vdash \exists x \forall y . \varphi(x, y) \iff \Phi \vdash \varphi(t_1, y_1) \vee \cdots \vee \varphi(t_n, y_n)$$

where t_i does not contain $y_i, \ldots, y_n$.

The general form requires the constant domain axiom (cd):

$$\forall x.(\alpha(x) \vee \beta) \leftrightarrow (\forall x.\alpha(x)) \vee \beta.$$

Its algebraic counterpart is meet infinite distributivity:

$$(mid) \qquad \bigwedge_{i \in I}(x_i \vee y) = (\bigwedge_{i \in I} x_i) \vee y.$$

Lemma

Let $\mathbf{A}$ be an FL algebra.

- If $\mathbf{A}$ is distributive, then $\mathbf{A}^\sigma$ satisfies (mid).
- If $\mathbf{A}$ is an MTL algebra, then $\mathbf{A}^h$ satisfies (mid).

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Theorem

Let $\mathbf{L}$ be a substructural logic. Herbrand's theorem for ∃∀-formulas holds for $\mathbf{QL}(cd)$ if

- *either* $\mathbf{L}$ is axiomatized by distributivity and some $\mathcal{N}_2$ axioms,
- *or* $\mathbf{L}$ is axiomatized by (e), (w), (pl) and and some $\mathcal{P}_3$ axioms.

Further topics

- Apparently there is no nice duality between FL algebras and residuated frames.
- Residuated frames are close to syntax so that one can encode syntactic information into frames.
- By encoding proof theoretic arguments into frames and taking the complex algebra, one can obtain an algebraic construction.

Let X, Y be sets of propositional variables.

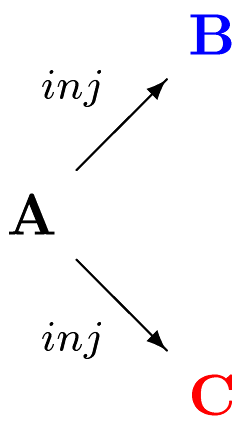
Maehara's lemma

If $\vdash_{\mathbf{FLe}} \Gamma, \Delta \Rightarrow \Pi$ with $\Gamma \subseteq Fm(X)$ and $\Delta, \Pi \subseteq Fm(Y)$, there is $\iota \in Fm(X \cap Y)$ such that

$$\vdash_{\mathbf{FLe}} \Gamma \Rightarrow \iota \quad \text{and} \quad \vdash_{\mathbf{FLe}} \iota, \Delta \Rightarrow \Pi.$$

Case study: Interpolation ⇒ Amalgamation

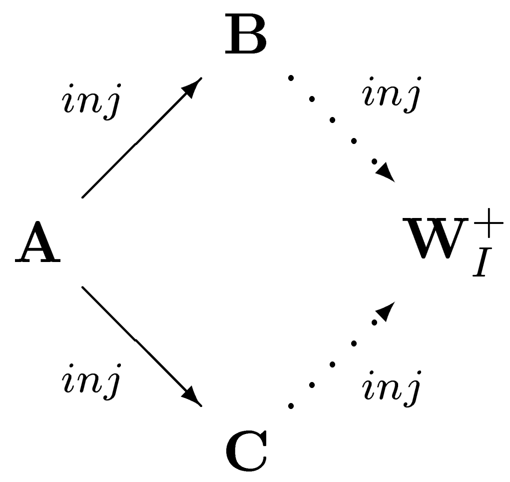
Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be FLe algebras.
Suppose that $\mathbf{A}$ is a subalgebra of both $\mathbf{B}$ and $\mathbf{C}$.



Define

$$
\begin{array}{rcl}
\mathbf{W}_I & := & (B \times C, B \cup C, N, \cdots) \\
(b, c)\ N\ c' & \Longleftrightarrow & \exists i \in A.\quad b \leq_{\mathbf{B}} i \quad \text{and} \quad ic \leq_{\mathbf{C}} c' \\
(b, c)\ N\ b' & \Longleftrightarrow & \exists i \in A.\quad c \leq_{\mathbf{C}} i \quad \text{and} \quad ib \leq_{\mathbf{B}} b'.
\end{array}
$$

Then the complex algebra gives rise to an amalgam.



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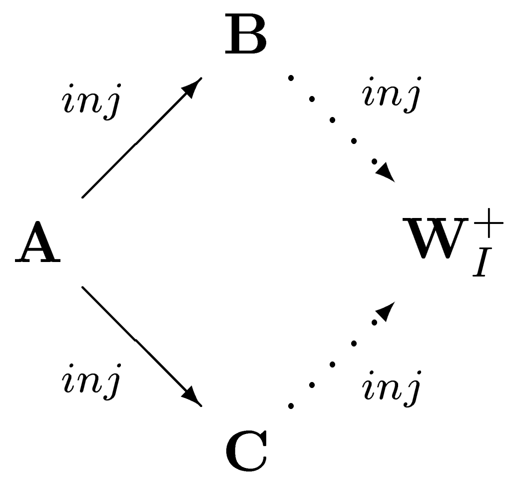

$$(\text{interpolation})^+ \quad = \quad \text{amalgamation}$$

Another success: density rule elimination $\Rightarrow$ densification

Likewise, the next talk by Horčík is an outcome of:

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Likewise, the next talk by Horčík is an outcome of:

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The slogan is:

$$\text{(proof theoretic argument)}^+ \quad = \quad \text{algebraic construction}$$

This way we can salvage nice proof theoretic ideas and bring them to ordered algebras.

Conclusion

We have explored the connection between proof theoretic arugments and algebraic completions based on the substructural hierarchy:

	sequent calc. $(\mathcal{N}_2)$	hypersequent calc. $(\mathcal{P}_3)$
cut elimination	MacNeille compl.	hyper-MacNeille compl.
Herbran's theorem	canonical ext.	hypercanonical ext.

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Substructural proof theory is full of bureacracy, but hopefully there are still something good to be salvaged.