# An axiomatization of iteration-free $PDL$ with loop

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## 1 Introduction

Propositional dynamic logic ($PDL$) is based on the idea of associating with each program $\gamma$ a modality $[\gamma]$, $[\gamma]\varphi$ being read "whenever $\gamma$ terminates it must do so in a state satisfying $\varphi$" [6]. Hence, $PDL$ is a modal logic with an algebraic structure in the set of modalities: composition $(\gamma;\delta)$, test $\varphi?$, union $(\gamma\cup\delta)$ and iteration $\gamma^\star$. Additional topics include results about axiomatization and decidability of $PDL$ variants. An interesting variant of $PDL$ is $PDL$ with loop [4]. Its chief feature is that loop of programs is not modally definable in the ordinary language of $PDL$ [9]. In this paper, we present the deductive system of iteration-free $PDL$ with loop.

## 2 Syntax and semantics

*Syntax* $p$ ranging over a countable set of propositional variables and $\pi$ ranging over a countable set of program variables, the set $FOR$ of all formulas ($\varphi$, $\psi$, etc) and the set $PRO$ of all programs ($\gamma$, $\delta$, etc) are defined as follows

$$\varphi ::= p \mid \bot \mid [\gamma]\varphi \mid \gamma!,$$

$$\gamma ::= \pi \mid (\gamma;\delta) \mid \varphi?.$$

The other constructs are defined as usual. In particular,

$$\neg\varphi ::= [\varphi?]\bot,$$

$$(\varphi\to\psi) ::= [\varphi?]\psi,$$

$$\langle\gamma\rangle\varphi ::= [[\gamma][\varphi?]\bot?]\bot.$$

We follow the standard rules for omission of the parentheses.

*Semantics* A model is a triple $(W,R,V)$ where $W\neq\emptyset$, $R:\ PRO\mapsto 2^{W\times W}$ and $V:\ W\mapsto 2^{FOR}$ are such that
**(i)** $\bot\notin V(x)$,
**(ii)** $[\gamma]\varphi\in V(x)$ iff for all $y\in W$, if $xR(\gamma)y$ then $\varphi\in V(y)$,
**(iii)** $\gamma!\in V(x)$ iff $xR(\gamma)x$,
**(iv)** $xR(\gamma;\delta)y$ iff there is $z\in W$ such that $xR(\gamma)z$ and $zR(\delta)y$,
**(v)** $xR(\varphi?)y$ iff $x=y$ and $\varphi\in V(y)$.
We say $\varphi$ is m-valid iff for all models $(W,R,V)$ and for all $x\in W$, $\varphi\in V(x)$.

## 3 Axiomatization

Let $f : PRO \mapsto PRO$ be defined by
**(i)** $f(\pi) = \pi$,
**(ii)** $f(\gamma; \delta) = f(\gamma); (\top?; f(\delta))$,
**(iii)** $f(\varphi?) = \varphi?$.
Let $\dim : PRO \mapsto \mathbb{N}$ be defined by
**(i)** $\dim(\pi) = 1$,
**(ii)** $\dim(\gamma; \delta) = \dim(\gamma) + \dim(\delta)$,
**(iii)** $\dim(\varphi?) = 0$.
Let $\equiv$ be the least equivalence relation on $PRO$ compatible with ; and such that $\gamma; (\delta; \lambda) \equiv (\gamma; \delta); \lambda$. Let $\preceq$ be the least reflexive transitive relation on $PRO$ containing $\equiv$, compatible with ; and such that
**(i)** if $\dim(\gamma) = 0$ then $\gamma; \delta \preceq \delta$,
**(ii)** if $\dim(\delta) = 0$ then $\gamma; \delta \preceq \gamma$,
**(iii)** $\gamma \preceq \gamma; \top?$,
**(iv)** $\delta \preceq \top?; \delta$,
**(v)** $\gamma!? \preceq \gamma$,
**(vi)** if $\dim(\gamma) = 0$ then $\gamma \preceq \gamma!?$,
**(vii)** $(\varphi \wedge \psi)? \preceq \varphi?; \psi?$.
Let $PDL_0^{loop}$ be the least normal logic that contains the axioms
$(\mathbf{A_1})$ $\langle\gamma; \delta\rangle\varphi \leftrightarrow \langle\gamma\rangle\langle\delta\rangle\varphi$,
$(\mathbf{A_2})$ if $\gamma \equiv \delta$ then $\langle\gamma\rangle\varphi \leftrightarrow \langle\delta\rangle\varphi$,
$(\mathbf{A_3})$ if $\dim(\gamma) = 0$ then $\langle\gamma\rangle\varphi \to \varphi$,
$(\mathbf{A_4})$ if $\gamma \preceq \delta$ then $\langle\gamma\rangle\varphi \to \langle\delta\rangle\varphi$,
$(\mathbf{A_5})$ $\langle\gamma(\varphi?)\rangle\chi \to \langle\gamma((\varphi \wedge \psi)?)\rangle\chi \vee \langle\gamma((\varphi \wedge \neg\psi)?)\rangle\chi$,
$(\mathbf{A_6})$ $\langle\gamma((\varphi \vee \psi)?)\rangle\chi \to \langle\gamma(\varphi?)\rangle\chi \vee \langle\gamma(\psi?)\rangle\chi$,
$(\mathbf{A_7})$ $\varphi \to \neg(\gamma; \neg(\delta; \varphi?; \gamma)!?; \delta)!$.
Obviously, every axiom is m-valid. Hence,

**Proposition 1.** *Let $\varphi \in FOR$. If $\varphi \in PDL_0^{loop}$ then $\varphi$ is m-valid.*

A special case of axiom $(\mathbf{A_4})$ is given by the formulas
$(\mathbf{A_4^k})$ $\langle\varphi_1?; \ldots; \varphi_k?\rangle\psi \to \langle(\varphi_1?; \ldots; \varphi_k?)!?\rangle\psi$
where $k \geq 1$. For all $n \in \mathbb{N}$, let $PDL_0^{loop|n}$ be the least normal logic that contains all axioms of $PDL_0^{loop}$ but the formulas $(\mathbf{A_4^k})$ where $k > n$. Obviously, $\bigcup\{PDL_0^{loop|n} : n \in \mathbb{N}\} = PDL_0^{loop}$. Moreover, for all $n \in \mathbb{N}$, one can find a $PDL_0^{loop|n}$-model in which $PDL_0^{loop|n+1}$ does not hold. Hence,

**Proposition 2.** *Axiom $(\mathbf{A_4})$ cannot be replaced by finitely many formulas.*

## 4 Theories, large programs and large systems

*Theories* A theory is any set of formulas containing $PDL_0^{loop}$ and closed under modus ponens. We say a theory $S$ is consistent iff $\bot \notin S$. We say a theory $S$ is maximal iff for all $\varphi \in FOR$, either $\varphi \in S$, or $\neg\varphi \in S$. Let $MAX$ be the set of all maximal theories.

By Lindenbaum's Lemma, for all $\varphi \in FOR$ and for all theories $S$, if $\varphi \notin S$ then there is $T \in MAX$ such that $S \subseteq T$ and $\varphi \notin T$. If $\gamma$ is a program and $S$ is a theory then let $[\gamma]S = \{\varphi\colon [\gamma]\varphi \in S\}$. The canonical model for $PDL_0^{loop}$ possesses all properties characterizing models but the third one, seeing that loop of programs is not modally definable in the ordinary language of $PDL$ [9]. Hence, following the line of reasoning suggested in [1, 2], the concept of large programs will be used.

*Large programs* For all theories $S$, let $S?$ be a new symbol. The set $LAR$ of all large programs ($\Gamma$, $\Delta$, etc) is defined by

$$\Gamma ::= \pi \mid (\Gamma; \Delta) \mid S?.$$

We say large program $\Gamma(S_1?, \ldots, S_n?)$ is maximal iff $S_1, \ldots, S_n \in MAX$. Let $\ker : LAR \mapsto 2^{PRO}$ be defined by
**(i)** $\ker(\pi) = \{\pi\}$,
**(ii)** $\ker(\Gamma; \Delta) = \{\gamma; \delta\colon \gamma \in \ker(\Gamma)$ and $\delta \in \ker(\Delta)\}$,
**(iii)** $\ker(S?) = \{\varphi?\colon \varphi \in S\} \cup \{\gamma\colon \gamma! \in S\}$.
Let $\dim :\ LAR \mapsto \mathbb{N}$ be defined by
**(i)** $\dim(\pi) = 1$,
**(ii)** $\dim(\Gamma; \Delta) = \dim(\Gamma) + \dim(\Delta)$,
**(iii)** $\dim(S?) = 0$.
If $\Gamma \in LAR$ and $S$ is a theory then let $[\Gamma]S = \{\varphi :\ \gamma \in \ker(\Gamma)$ and $[\gamma]\varphi \in S\}$. Let $\equiv$ be the binary relation on $LAR$ such that $\Gamma \equiv \Delta$ iff

- for all $\gamma \in \ker(\Gamma)$, if $\dim(\gamma) = \dim(\Gamma)$ then there is $\delta \in \ker(\Delta)$ such that $\dim(\delta) = \dim(\Delta)$ and $\gamma \equiv \delta$,
- for all $\delta \in \ker(\Delta)$, if $\dim(\delta) = \dim(\Delta)$ then there is $\gamma \in \ker(\Gamma)$ such that $\dim(\gamma) = \dim(\Gamma)$ and $\gamma \equiv \delta$.

Let $\preceq$ be the binary relation on $LAR$ such that $\Gamma \preceq \Delta$ iff

- for all $\delta \in \ker(\Delta)$, if $\dim(\delta) = \dim(\Delta)$ then there is $\gamma \in \ker(\Gamma)$ such that $\dim(\gamma) = \dim(\Gamma)$ and $\gamma \preceq \delta$.

*Large systems* A large system is a triple $(W, R, V)$ where $W \neq \emptyset$ and $R :\ LAR \mapsto 2^{W \times W}$ and $V :\ W \mapsto MAX$ are such that
**(i)** $\bot \notin V(x)$,
**(ii)** $[\gamma]\varphi \in V(x)$ iff for all $y \in W$, if there is a maximal $\Gamma \in LAR$ such that $f(\gamma) \in \ker(\Gamma)$ and $xR(\Gamma)y$ then $\varphi \in V(y)$,
**(iii)** $\gamma! \in V(x)$ iff there is a maximal $\Gamma \in LAR$ such that $f(\gamma) \in \ker(\Gamma)$ and $xR(\Gamma)x$,
**(iv)** $R(\Gamma; S?; \Delta) = \{(x, y)\colon$ there is $z \in W$ such that $xR(\Gamma)z$, $S \subseteq V(z)$ and $zR(\Delta)y\}$,
**(v)** $R(\Gamma; S?) = \{(x, y)\colon xR(\Gamma)y$ and $S \subseteq V(y)\}$,
**(vi)** $R(S?; \Delta) = \{(x, y)\colon S \subseteq V(x)$ and $xR(\Delta)y\}$,
**(vii)** $R(S?) = \{(x, y)\colon x = y$ and $S \subseteq V(y)\}$,
**(viii)** if $\Gamma \preceq \Delta$ then $R(\Gamma) \subseteq R(\Delta)$.
We say $\varphi$ is ls-valid iff for all large systems $(W, R, V)$ and for all $x \in W$, $\varphi \in V(x)$. Obviously, every large system corresponds to a model. Hence,

**Proposition 3.** *Let $\varphi \in FOR$. If $\varphi$ is m-valid then $\varphi$ is ls-valid.*

## 5 Subordination models

A subordination model is a triple $(W, R, V)$ where $W \neq \emptyset$ and $R: LAR \mapsto 2^{W \times W}$ and $V: W \mapsto MAX$ are such that
**(i)** $\bot \notin V(x)$,
**(ii)** if $[\gamma]\varphi \in V(x)$ then for all $y \in W$, if there is a maximal $\Gamma \in LAR$ such that $f(\gamma) \in \ker(\Gamma)$ and $xR(\Gamma)y$ then $\varphi \in V(y)$,
**(iii)** $\gamma! \in V(x)$ iff there is a maximal $\Gamma \in LAR$ such that $f(\gamma) \in \ker(\Gamma)$ and $xR(\Gamma)x$,
**(iv)** $R(\Gamma; S?; \Delta) \supseteq \{(x, y)$: there is a $z \in W$ such that $xR(\Gamma)z$, $S \subseteq V(z)$ and $zR(\Delta)y\}$,
**(v)** $R(\Gamma; S?) = \{(x, y)$: $xR(\Gamma)y$ and $S \subseteq V(y)\}$,
**(vi)** $R(S?; \Delta) = \{(x, y)$: $S \subseteq V(x)$ and $xR(\Delta)y\}$,
**(vii)** $R(S?) = \{(x, y)$: $x = y$ and $S \subseteq V(y)\}$,
**(viii)** if $\Gamma \preceq \Delta$ then $R(\Gamma) \subseteq R(\Delta)$.
We say $\varphi$ is sm-valid iff for all subordination models $(W, R, V)$ and for all $x \in W$, $\varphi \in V(x)$. Obviously, for all consistent $S \in MAX$, the triple $(W, R, V)$ where $W = \{S\}$, $SR(\Gamma)S$ iff $S? \preceq \Gamma$ and $V(S) = S$ is a subordination model. Hence,

**Proposition 4.** *Let $\varphi \in FOR$. If $\varphi$ is sm-valid then $\vdash_{PDL_0^{loop}} \varphi$.*

Given a subordination model $(W, R, V)$, it may contain imperfections:
**(i)** triples $(\gamma, \varphi, x)$ where $\gamma \in PRO$, $\varphi \in FOR$ and $x \in W$ are such that $[\gamma]\varphi \notin V(x)$ and for all $y \in W$, if there is a maximal $\Gamma \in LAR$ such that $f(\gamma) \in \ker(\Gamma)$ and $xR(\Gamma)y$ then $\varphi \in V(y)$,
**(ii)** 5-tuples $(\Gamma, S, \Delta, x, y)$ where $\Gamma, \Delta \in LAR$ are maximal, $S \in MAX$ and $x, y \in W$ are such that $xR(\Gamma; S?; \Delta)y$ and for all $z \in W$, either $x\overline{R(\Gamma)}z$, or $S \not\subseteq V(z)$, or $z\overline{R(\Delta)}y$.
An imperfection $(\gamma, \varphi, x)$ can be repaired by adding a new element $y$ to $W$ and by extending the functions $R$ and $V$ in such a way that $y$ will be $\gamma$-reachable from $x$ and $y$ will not satisfy $\varphi$ whereas an imperfection $(\Gamma, S, \Delta, x, y)$ can be repaired by adding a new element $z$ to $W$ and by extending the functions $R$ and $V$ in such a way that $z$ will be $\Gamma$-reachable from $x$, $y$ will be $\Delta$-reachable from $z$ and $z$ will satisfy $S$. The heart of our method consists in step-by-step repairing all these imperfections, therefore transforming every subordination model into an equivalent large system. Hence,

**Proposition 5.** *Let $\varphi \in FOR$. If $\varphi$ is ls-valid then $\varphi$ is sm-valid.*

From Proposition 1 and Propositions 3–5, we obtain the following

**Theorem 1.** *Let $\varphi \in FOR$. The following conditions are equivalent:*
**(i)** $\vdash_{PDL_0^{loop}} \varphi$,
**(ii)** *$\varphi$ is m-valid,*
**(iii)** *$\varphi$ is ls-valid,*
**(iv)** *$\varphi$ is sm-valid.*

## References


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