Two Neighborhood Semantics for Subintuitionistic Logics

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In [7] we defined two neighborhood semantics for subintuitionistic logics. NB-semantics, our main semantics, is for most purposes best suitable to study the basic logic and its extensions. The N-semantics is closer to the usual neighborhood semantics for modal logics, and is thereby more suitable to study Gödel-type translations into modal logics. The relationship between the two semantics remained unclear. Our basic logic WF is sound and complete for NB-semantics and sound for N-semantics but completeness remained an open issue. Here we clear up their relationship. We introduce a new rule N, which added to WF gives a system $\mathsf{WF_N}$ complete for N-semantics. Two new axioms, falsifiable in NB-semantics, can be derived from it. Gödel-type translations into modal logic can now be realized properly.

Definition 1. *$\mathfrak{F}=\langle W,g,NB,\mathcal{X}\rangle$ is called an **NB-Neighborhood Frame** of subintuitionistic logic if $W\neq\emptyset$ and $\mathcal{X}$ is a non-empty collection of subsets of W such that $\emptyset$ and W belong to $\mathcal{X}$, and $\mathcal{X}$ is closed under $\cup$, $\cap$ and $\to$ defined by*

$$U\to V := \{w\in W \mid (U,V)\in NB(w)\},$$

where NB is a function from W into $\mathcal{P}(\mathcal{X}^2)$ such that:

1. $\forall w\in W,\ \forall X,Y\in\mathcal{X},\ (X\subseteq Y \ \Rightarrow\ (X,Y)\in NB(w))$,
2. $NB(g)=\{(X,Y)\in\mathcal{X}^2 \mid X\subseteq Y\}$ *(g is called **omniscient**).*

*In an **NB-Neighborhood Model** $\mathfrak{M}=\langle W,g,NB,\mathcal{X},V\rangle$, $V: At\to\mathcal{X}$ is a valuation function on the set of propositional variables At.*

***Truth** of A in w, $w\Vdash A$ is defined as usual except for: $\mathfrak{M},w\Vdash A\to B \ \Leftrightarrow\ (A^{\mathfrak{M}},B^{\mathfrak{M}})\in NB(w)$, where $A^{\mathfrak{M}}:=\{w\in W \mid \mathfrak{M},w\Vdash A\}$.*

Definition 2. *$\mathfrak{F}=\langle W,g,N,\mathcal{X}\rangle$ is an **N-Neighborhood Frame** if W is a non-empty set and $\mathcal{X}$ is a non-empty collection of subsets of W such that $\emptyset$ and W belong to $\mathcal{X}$ and $\mathcal{X}$ is closed under $\cup$, $\cap$ and $\to$ defined by*

$$U\to V := \{w\in W \mid \overline{U}\cup V\in N(w)\},$$

*where N is a function from W into $\mathcal{P}(\mathcal{X})$, $g\in W$, for each $w\in W$, $W\in N(w)$, $N(g)=\{W\}$ (g is called **omniscient**). Valuation $V: At\to\mathcal{X}$ makes $\mathfrak{M}=\langle W,g,N,\mathcal{X},V\rangle$ an **N-Neighborhood Model** with the clause:*

$$\mathfrak{M},w\Vdash A\to B \Leftrightarrow \{v \mid v\Vdash A \Rightarrow v\Vdash B\} = \overline{A^{\mathfrak{M}}}\cup B^{\mathfrak{M}}\in N(w).$$

Definition 3. WF *is the logic given by the following axioms and rules,*

1. $A\to A\vee B$	2. $B\to A\vee B$	3. $A\to A$
4. $A\wedge B\to A$	5. $A\wedge B\to B$	6. $\frac{A \quad A\to B}{B}$
7. $\frac{A\to B \quad A\to C}{A\to B\wedge C}$	8. $\frac{A\to C \quad B\to C}{A\vee B\to C}$	9. $\frac{A\to B \quad B\to C}{A\to C}$
10. $\frac{A}{B\to A}$	11. $\frac{A\leftrightarrow B \quad C\leftrightarrow D}{(A\to C)\leftrightarrow(B\to D)}$	12. $\frac{A \quad B}{A\wedge B}$
13. $A\wedge(B\vee C)\to(A\wedge B)\vee(A\wedge C)$		14. $\bot\to A$

To the system WF *we add the rule* N *to obtain the logic* $\mathsf{WF_N}$*:*

$$\frac{C\to A\vee D \qquad A\wedge C\wedge B\to D}{(A\to B)\to(C\to D)} \qquad (\mathsf{N})$$

A rule like N is considered to be valid on a frame $\mathfrak{F}$ if, on each $\mathfrak{M}$ on which the premises of the rule are valid, the conclusion is valid as well.

Lemma 1. *(Soundness of* $\mathsf{WF_N}$*)* N *is valid on* N*-neighborhood frames.*

Proof. Recall that, by Theorem 2.13(1) of [7], for all E, F, $\mathfrak{M} \Vdash E \to F$ iff $E^{\mathfrak{M}} \subseteq F^{\mathfrak{M}}$.

Assume, (1) $\mathfrak{M} \Vdash C \to A \vee D$, i.e. $C^{\mathfrak{M}} \subseteq A^{\mathfrak{M}} \cup D^{\mathfrak{M}}$, and (2) $\mathfrak{M} \Vdash A \wedge C \wedge B \to D$, i.e. $A^{\mathfrak{M}} \cap C^{\mathfrak{M}} \cap B^{\mathfrak{M}} \subseteq D^{\mathfrak{M}}$. It will suffice to prove that $\overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}} \subseteq \overline{C^{\mathfrak{M}}} \cup D^{\mathfrak{M}}$.

Let $w \in \overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}}$. Then $w \in \overline{A^{\mathfrak{M}}}$ or ($w \in A^{\mathfrak{M}}$ and $w \in B^{\mathfrak{M}}$). If $w \in \overline{A^{\mathfrak{M}}}$, we distinguish the cases $w \in D^{\mathfrak{M}}$ and $w \in \overline{D^{\mathfrak{M}}}$. In the first case we are done directly. In the second case, we can conclude from (1) that $w \in \overline{C^{\mathfrak{M}}}$ and we are done as well. If $w \in A^{\mathfrak{M}}$ and $w \in B^{\mathfrak{M}}$, we distinguish the cases $w \in \overline{C^{\mathfrak{M}}}$ and $w \in C^{\mathfrak{M}}$. In the first case we are done directly. In the second case, we can conclude from (2) that $w \in D^{\mathfrak{M}}$ and we are done as well.

□

Definition 4. *A set of sentences* Δ *is a* ***prime theory*** *if and only if*

- $A, B \in \Delta \;\Rightarrow\; A \wedge B \in \Delta$,
- $\vdash A \to B$ and $A \in \Delta \;\Rightarrow\; B \in \Delta$,
- $\vdash A \;\Rightarrow\; A \in \Delta$,
- $A \vee B \in \Delta \;\Rightarrow\; A \in \Delta$ or $B \in \Delta$.

Lemma 2. $\mathsf{WF_N}$ *is a prime theory (has the disjunction property).*

Proof. Using Kleene's $|$ ([6]) as in [7], Theorem 2.12. □

Definition 5. *Let* $W_{\mathsf{WF_N}}$ *be the set of all consistent prime theories of* $\mathsf{WF_N}$*. Given a formula* A*, we define* $\llbracket A \rrbracket = \{\Delta \mid \Delta \in W_{\mathsf{WF_N}},\; A \in \Delta\}$*. The* ***N-Canonical model*** $\mathfrak{M}_{\mathsf{WF_N}} = \langle W, g, N, \mathcal{X}, V\rangle$ *is defined by:*

- $W = W_{\mathsf{WF_N}}$,
- $g = \mathsf{WF_N}$,
- *For each* $\Gamma \in W$, $N(\Gamma) = \{\overline{\llbracket A \rrbracket} \cup \llbracket B \rrbracket \mid A \to B \in \Gamma\}$,
- $\mathcal{X}$ *is the set of all* $\llbracket A \rrbracket$,
- *If* $p \in At$*, then* $V(p) = \llbracket p \rrbracket = \{\Gamma \mid \Gamma \in W \text{ and } p \in \Gamma\}$.

Lemma 3. (Truth Lemma) *In the N-canonical Model* $\mathfrak{M}_{\mathsf{WF_N}}$, $A \in \Gamma$ *iff* $\Gamma \Vdash A$.

Proof. The crucial part of the proof is showing that, if $\overline{\llbracket A \rrbracket} \cup \llbracket B \rrbracket = \overline{\llbracket C \rrbracket} \cup \llbracket D \rrbracket$, then $\mathsf{WF_N} \vdash (A \to B) \leftrightarrow (C \to D)$. So, assume $\overline{\llbracket A \rrbracket} \cup \llbracket B \rrbracket = \overline{\llbracket C \rrbracket} \cup \llbracket D \rrbracket$. It suffices to show (1) $\mathsf{WF_N} \vdash A \to B \vee C$, $\mathsf{WF_N} \vdash A \wedge C \wedge D \to B$ and (2) $\mathsf{WF_N} \vdash C \to A \vee D$, $\mathsf{WF_N} \vdash A \wedge C \wedge B \to D$. We will show (1); (2) is analogous.

From $\overline{\llbracket A \rrbracket} \cup \llbracket B \rrbracket = \overline{\llbracket C \rrbracket} \cup \llbracket D \rrbracket$ we get $\llbracket A \rrbracket \cap \overline{\llbracket B \rrbracket} = \llbracket C \rrbracket \cap \overline{\llbracket D \rrbracket}$. We have $\llbracket A \rrbracket \subseteq \llbracket B \rrbracket \cup \llbracket A \rrbracket$, so also, $\llbracket A \rrbracket \subseteq \llbracket B \rrbracket \cup (\llbracket A \rrbracket \cap \overline{\llbracket B \rrbracket})$, This means that $\llbracket A \rrbracket \subseteq \llbracket B \rrbracket \cup (\llbracket C \rrbracket \cap \overline{\llbracket D \rrbracket})$, so $\llbracket A \rrbracket \subseteq \llbracket B \rrbracket \cup \llbracket C \rrbracket$. Therefore, $A \to B \vee C \in g$, so $\mathsf{WF_N} \vdash A \to B \vee C$.

Again using $\llbracket A \rrbracket \cap \overline{\llbracket B \rrbracket} = \llbracket C \rrbracket \cap \overline{\llbracket D \rrbracket}$, we get $\llbracket A \rrbracket \cap \llbracket C \rrbracket \cap \llbracket D \rrbracket \cap \overline{\llbracket B \rrbracket} = \llbracket A \rrbracket \cap \overline{\llbracket B \rrbracket} \cap \llbracket C \rrbracket \cap \llbracket D \rrbracket = \llbracket C \rrbracket \cap \overline{\llbracket D \rrbracket} \cap \llbracket C \rrbracket \cap \llbracket D \rrbracket = \emptyset$. So, $\llbracket A \rrbracket \cap \llbracket C \rrbracket \cap \llbracket D \rrbracket \subseteq \llbracket B \rrbracket$, and, reasoning as above, $\mathsf{WF_N} \vdash A \wedge C \wedge D \to B$.

□

Theorem 1. (Completeness of $\mathsf{WF_N}$) $\Sigma \vdash_{\mathsf{WF_N}} A$ *iff for all* $w \in \mathfrak{M}_{\mathsf{WF_N}}$*, if* $w \Vdash \Sigma$*, then* $w \Vdash A$.

Lemma 4. $\mathsf{WF_N} \vdash (A \to B) \leftrightarrow (A \vee B \to B)$, $\quad \mathsf{WF} \nvdash (A \to B) \leftrightarrow (A \vee B \to B)$.
$\mathsf{WF_N} \vdash (A \to B) \leftrightarrow (A \to A \wedge B)$, $\quad \mathsf{WF} \nvdash (A \to B) \leftrightarrow (A \to A \wedge B)$.

The exact relationship between the axioms of Lemma 4 and rule N is unclear. We can derive the axioms of Lemma 4 from WF+N, but the other direction seems unlikely, probably N is not derivable from WF + the axioms of Lemma 4.

We can now extend the translation results of Corsi [3] and others [4, 1] on subintuitionistic logics into modal logics to weaker logics. We consider the translation $\Box$ from $\mathcal{L}$, the language of IPC, to $\mathcal{L}_\Box$, the language of modal propositional logic. It is given by:

1. $p^{\Box} = p$;
2. $(A \wedge B)^{\Box} = A^{\Box} \wedge B^{\Box}$;
3. $(A \vee B)^{\Box} = A^{\Box} \vee B^{\Box}$;
4. $(A \to B)^{\Box} = \Box(A^{\Box} \to B^{\Box})$.

Theorem 2. *For all formulas A, $\mathsf{WF_N} \vdash A$ iff $\mathsf{EN} \vdash A^{\Box}$.*
For all formulas A, $\mathsf{WF_N I_R I_L} \vdash A$ iff $\mathsf{M} \vdash A^{\Box}$.

Here *classical modal logic* E, based on $\frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$, is the smallest non-normal modal logic, and EN extends E by adding necessitation. Also a system of modal logic is *monotonic* iff it is closed under RM ($\frac{A \to B}{\Box A \to \Box B}$), and M is the smallest monotonic modal logic [2, 5]. In [7], $\mathsf{I_L}$ is the rule $\frac{A \to B}{(C \to A) \to (C \to B)}$ and $\mathsf{I_R}$ is the rule $\frac{A \to B}{(B \to C) \to (A \to C)}$. In the meantime we have been able to show that rule $\mathsf{I_L}$ is equivalent to the axiom $\check{\mathsf{C}}$: $(A \to B \wedge C) \to (A \to B) \wedge (A \to C)$, and rule $\mathsf{I_R}$ to the axiom $\check{\mathsf{D}}$: $(A \vee B \to C) \to (A \to C) \wedge (B \to C)$.

In [7] the relationship between the logic WF and the non-normal modal logic EN was already indicated. But because of the difference of the models we were able to prove only the direction $\vdash_{\mathsf{WF}} A \Rightarrow \vdash_{\mathsf{EN}} A$. A similar situation arose between the basic monotonic modal logic M and our system $\mathsf{WFI_R I_L}$.

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