AN EHRENFEUCHT-FRAÏSSÉ GAME FOR INQUISITIVE FIRST-ORDER LOGIC

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Abstract. Inquisitive first-order logic, InqBQ, extends classical first-order logic with questions. From a technical perspective, InqBQ allows us to talk about a plurality of first-order structures, expressing not only facts about each structure in isolation, but also about how these structures relate to each other. We describe an Ehrenfeucht-Fraïssé game for InqBQ and show that this characterizes the expressive power of the logic. We illustrate the usefulness of the result by giving a game-theoretic proof of the fact that certain cardinality properties are not expressible in InqBQ.

Keywords: inquisitive logic; Ehrenfeucht-Fraïssé games; team semantics; model theory

Background: inquisitive first-order logic

Inquisitive first-order logic, InqBQ, ([2], [6], [4]) generalizes classical first-order logic to interpret not only formulas that stand for statements, but also formulas expressing questions and dependencies. Technically, InqBQ fits within the family of logics based on *team semantics*, being closely related to Dependence Logic ([7], [1]; for a discussion of this connection, see [3], [8]). A model for InqBQ represents a variety of states of affairs, or *worlds*, where each world corresponds to a first-order structure. While standard first-order formulas only express local requirements, which have to be satisfied at each world in the model, inquisitive formulas (aka questions) allow us to express global requirements, having to do with the way the worlds are related to each another. Thus, for instance, in InqBQ we can express that there is one individual that, uniformly across all the worlds in our model, has property P; or that the extension of property P is the same in all worlds in the model; or that, within the model, the extension of property Q is functionally determined by the extension of property P.

Let us recall here the essential definitions. In this paper, we assume that the given signature is finite and relational, that is, contains no function symbols. While the finiteness requirement is essential for our result to hold, the relationality requirement can be dropped. The syntax of InqBQ is given by the following inductive definition.

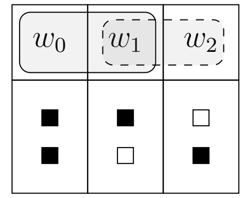
$$\varphi \quad ::= \quad R(x_1,\dots,x_n) \mid (x=y) \mid \bot \mid \varphi\wedge\varphi \mid \varphi\to\varphi \mid \forall x.\varphi \mid \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \varphi \mid \overline{\exists} x.\varphi$$

$$\begin{array}{lcl}
\mathcal{M},s\vDash_g R(\overline{x}) & \iff & \forall w\in s.\ g(\overline{x})\in I_w(R)\\
\mathcal{M},s\vDash_g [x=y] & \iff & \forall w\in s.\ g(x)\sim_w g(y)\\
\mathcal{M},s\vDash_g \bot & \iff & s=\emptyset\\
\mathcal{M},s\vDash_g \varphi\wedge\psi & \iff & \mathcal{M},s\vDash_g\varphi \text{ and } \mathcal{M},s\vDash_g\psi\\
\mathcal{M},s\vDash_g \varphi\to\psi & \iff & \forall t\subseteq s.\,[\mathcal{M},t\vDash_g\varphi\Rightarrow\mathcal{M},t\vDash_g\psi]\\
\mathcal{M},s\vDash_g \forall x.\varphi & \iff & \forall d\in D^{\mathcal{M}}.\ \mathcal{M},s\vDash_{g[x\mapsto d]}\varphi\\
\mathcal{M},s\vDash_g \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi & \iff & \mathcal{M},s\vDash_g\varphi \text{ or } \mathcal{M},s\vDash_g\psi\\
\mathcal{M},s\vDash_g \overline{\exists} x.\varphi & \iff & \exists d\in D^{\mathcal{M}}.\ \mathcal{M},s\vDash_{g[x\mapsto d]}\varphi
\end{array}$$

Classical first-order logic can be identified with the $\{\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}, \overline{\exists}\}$-free fragment of the language (the operators $\neg, \vee, \exists$ can be defined in the usual way). A model for InqBQ is a tuple $\mathcal{M}=\langle W, D, I, \sim\rangle$ where W is a set (the worlds), D is a non-empty set (the individuals), and:

• I assigns to each world $w\in W$ a first-order structure I_w over the domain D; we will denote by $I_w(R)$ the interpretation of the relation symbol R in I_w.

• $\sim$ assigns to each world $w\in W$ an equivalence relation $\sim_w$ over D (the identity relation at w), with the requirement that the relations and functions respect the classes of $\sim_w$.



We refer to a set $s \subseteq W$ as an *information state*. Intuitively, we can regard s as encoding the information that the actual state of affairs corresponds to one of the worlds in s. An assignment is a map $g : \texttt{Var} \to D$. The semantics of `InqBQ` is then given by a recursive definition of the relation of *support* relative to an information state $s \subseteq W$ and an assignment g as in the table in page 1. We write $\mathcal{M} \models_g \varphi$ for $\mathcal{M}, W \models_g \varphi$. In the picture, a representation of a simple model $\mathcal{M}$ in the signature $\{P^{(1)}\}$ is depicted: it has three worlds (w_0, w_1, w_2) and two elements; under each w_i, a copy of the domain is shown, together with the interpretation of property P (black squares); $\sim$ is the actual identity relation at each world. We have $\{w_1, w_2\} \vDash \exists x.P(x)$ but $\{w_1, w_2\} \not\vDash \overline{\exists} x.P(x)$: although in state $\{w_1, w_2\}$ we know that some individual with property P exists, we do not know of any individual that has the property P. By contrast, we have $\{w_0, w_1\} \models \overline{\exists} x.P(x)$, since in the state $\{w_0, w_1\}$ it is known that the upper individual has property P.

An Ehrenfeucht-Fraïssé game for `InqBQ`

Ehrenfeucht-Fraïssé games ([5]) have established themselves as one of the most useful model-theoretic tools to check if two models for a given logic are distinguishable by sentences of the logic itself. While these games were first developed for first-order logic, they have since been extended to a number of settings, including monadic second-order logic and modal logic. In this paper, we propose an Ehrenfeucht-Fraïssé game for `InqBQ`, which allows us to provide a game-theoretic characterization of equivalence with respect to `InqBQ` formulas.

Fix $\mathcal{M}$ and $\mathcal{N}$. The game can be thought as a dialog between two players, **I** (the *spoiler*) and **II** (the duplicator). **I** tries to show that $\mathcal{M}$ supports some information that $\mathcal{N}$ does not support by picking every turn an element or an info state that witness this fact. The role of **II** is to respond accordingly, simulating the choices of **I** and thus showing that such an info cannot exist.

Definition 1 (The game)**.** Consider two tuples $\langle \mathcal{M}_1, s_1, \overline{a}_1 \rangle$ and $\langle \mathcal{M}_2, s_2, \overline{a}_2 \rangle$ where $\mathcal{M}_1, \mathcal{M}_2$ are information models, s_1, s_2 are info states in the corresponding models, and $\overline{a}_1, \overline{a}_2$ are two tuples of individuals from the corresponding domains of the same length. For $m, n \in \mathbb{N}$ we define a zero-sum game $\mathrm{EF}_{m,n}(M_1, s_1, \overline{a}_1; M_2, s_2, \overline{a}_2)$, played between two players, **I** (the *spoiler*) and **II** (the *duplicator*). The game is defined inductively on the pair $\langle m, n \rangle$ as follows:

- Base case: $\langle m, n \rangle = \langle 0, 0 \rangle$. No move is performed and the game ends. Player **II** wins iff for every atomic $\varphi(\overline{x})$ with $\texttt{length}(\overline{x}) = \texttt{length}(\overline{a}_i)$: $\mathcal{M}_1, s_1 \vDash \varphi(\overline{a}_1) \;\Rightarrow\; \mathcal{M}_2, s_2 \vDash \varphi(\overline{a}_2)$
- Inductive case: **I** moves and **II** must respond accordingly. The following three options are allowed:

$\forall$ **move:** this move is allowed only if $n > 0$. **I** chooses $d_2 \in D^{\mathcal{M}_2}$ and **II** chooses $d_1 \in D^{\mathcal{M}_1}$. The game $\mathrm{EF}_{m,n-1}(\mathcal{M}_1, s_1, \overline{a}_1 d_1; \mathcal{M}_2, s_2, \overline{a}_2 d_2)$ starts. To win, **II** must win this game.

$\overline{\exists}$ **move:** this move is allowed only if $n > 0$. **I** chooses $d_1 \in D^{\mathcal{M}_1}$ and **II** chooses $d_2 \in D^{\mathcal{M}_2}$. The game $\mathrm{EF}_{m,n-1}(\mathcal{M}_1, s_1, \overline{a}_1 d_1; \mathcal{M}_2, s_2, \overline{a}_2 d_2)$ starts. To win, **II** must win this game.

$\to$ **move:** this move is allowed only if $m > 0$. **I** chooses $s_2' \subseteq s_2$ and **II** chooses $s_1' \subseteq s_1$. Then the games $\mathrm{EF}_{m-1,n}(\mathcal{M}_1, s_1', \overline{a}_1; \mathcal{M}_2, s_2', \overline{a}_2)$ and $\mathrm{EF}_{m-1,n}(\mathcal{M}_2, s_2', \overline{a}_2; \mathcal{M}_1, s_1', \overline{a}_1)$ start. In order to win, **II** must win both these games.

We indicate with $\mathcal{M}_1, s_1, \overline{a}_1 \preceq_{m,n} \mathcal{M}_2, s_2, \overline{a}_2$ that player **II** has a **winning strategy** in the game $\mathrm{EF}_{m,n}(M_1, s_1, \overline{a}_1; M_2, s_2, \overline{a}_2)$.

To connect the game to the expressivity of `InqBQ`-formulas, we introduce a notion of complexity which takes into account not only nesting of quantifiers, but also nesting of implications.

Definition 2 ($\texttt{IQ}$ complexity)**.** The $\texttt{I}$ and $\texttt{Q}$ complexities of a formula φ are defined in the table in page 2. Moreover, we define $\texttt{IQ}(\varphi) = \langle \texttt{I}(\varphi), \texttt{Q}(\varphi)\rangle$ and $\mathcal{L}^l_{m,n}$ the set of formulas of $\texttt{IQ}$ complexity at most $\langle m, n\rangle$ (with respect to the component-wise order) with free variables among $\{x_1, \dots, x_l\}$.

$\texttt{I}(A) = 0$	$\texttt{Q}(A) = 0$
$\texttt{I}(\psi \circ \chi) = \max(\texttt{I}(\psi), \texttt{I}(\chi))$	$\texttt{Q}(\psi \circ \chi) = \max(\texttt{Q}(\psi), \texttt{Q}(\chi))$
$\texttt{I}(\psi \to \chi) = \max(\texttt{I}(\psi), \texttt{I}(\chi)) + 1$	$\texttt{Q}(\psi \to \chi) = \max(\texttt{Q}(\psi), \texttt{Q}(\chi))$
$\texttt{I}(\Pi x.\psi) = \texttt{I}(\psi)$	$\texttt{Q}(\Pi x.\psi) = \texttt{Q}(\psi) + 1$

For A atomic, $\circ \in \{\wedge, \vee\}$ and $\Pi \in \{\overline{\exists}, \forall\}$

Theorem 3 (Ehrenfeucht-Fraïssé theorem for $\texttt{InqBQ}$)**.** *For $l = \texttt{length}(\overline{a}_i)$, we have:*

$$\mathcal{M}_1, s_1, \overline{a}_1 \preceq_{m,n} \mathcal{M}_2, s_2, \overline{a}_2 \iff \forall \varphi(\overline{x}) \in \mathcal{L}^l_{m,n}. [\mathcal{M}_1, s_1 \vDash \varphi(\overline{a}_1) \Rightarrow \mathcal{M}_2, s_2 \vDash \varphi(\overline{a}_2)]$$

Proof sketch. ($\Rightarrow$) By contraposition: let $\varphi \in \mathcal{L}^l_{m,n}$ be such that $\mathcal{M}_1, s_1 \vDash \varphi(\overline{a}_1)$ but $\mathcal{M}_2, s_2 \nvDash \varphi(\overline{a}_2)$. We will indicate this condition as $\mathfrak{I}(\varphi)$. We define recursively a winning strategy for **I** by maintaining the invariant $\mathfrak{I}$ for a subformula ψ of φ of suitable $\texttt{IQ}$ complexity during a run of the game. In particular, if the subgame has index $\langle m, n\rangle$ then $\texttt{IQ}(\psi) \leq \langle m, n\rangle$ (componentwise), assuring that ψ is atomic at the end of the game. The non-trivial cases are the following:
If $\varphi \equiv \forall x.\psi$, consider $d_2 \in D^{\mathcal{M}_2}$ such that $\mathcal{M}_2, s_2 \nvDash \psi(\overline{a}_2, d_2)$. Then by performing a $\forall$ move and by choosing d_2, **I** ensures that the condition $\mathfrak{I}(\psi)$ will hold in the next turn. With a similar argument we also obtain the result for $\varphi \equiv \overline{\exists} x.\psi$.
If $\varphi \equiv \psi \to \chi$, there exists an info state $s_2' \subseteq s_2$ s.t. $\mathcal{M}_2, s_2' \vDash \psi(\overline{a}_2)$ but $\mathcal{M}_2, s_2' \nvDash \chi(\overline{a}_2)$. Note that for each $s_1' \subseteq s_1$, one of the two corresponding relations fails. Then by performing a $\to$ move and by choosing s_2', **I** ensures that one of the conditions $\mathfrak{I}(\psi)$ and $\mathfrak{I}(\chi)$ holds for one of the subgames.
($\Leftarrow$) Again by contraposition: if **II** doesn't have a winning strategy, then by Gale-Stewart theorem, **I** does. By well-founded induction on $\langle m, n\rangle$ we can define φ for which condition $\mathfrak{I}(\varphi)$ holds.
Suppose the strategy of **I** starts with a $\forall$ move by choosing $d_2 \in D^{\mathcal{M}_2}$. Then by inductive hypothesis, for every $d_1 \in D^{\mathcal{M}_1}$ a formula $\psi_{d_1} \in \mathcal{L}^{l+1}_{m,n-1}$ can be found for which $\mathfrak{I}(\psi_{d_1})$ holds. From this we obtain $\mathfrak{I}(\varphi)$ for the following φ. For an $\overline{\exists}$ move, the argument is analogous.

$$\varphi :\equiv \forall y. \bigvee\!\!\!\!\bigvee \left\{\psi(\overline{x}, y) \in \mathcal{L}^{l+1}_{m,n-1} | \mathcal{M}_2, s_2 \nvDash \psi(\overline{a}_2, d_2)\right\}$$

If the strategy of **I** starts with a $\to$ move, by reasoning as in the previous case we can find a state $s_2' \subseteq s_2$ such that $\mathfrak{I}(\varphi)$ holds for the following φ:

$$X := \{\psi \in \mathcal{L}^l_{m-1,n} | \mathcal{M}_2, s_2' \vDash \psi(\overline{a}_2)\} \qquad \varphi :\equiv \bigwedge X \to \bigvee\!\!\!\!\bigvee (\mathcal{L}^l_{m-1,n} \setminus X)$$

Note that, since our signature is finite, the class $\mathcal{L}^l_{m-1,n}$ contains only finitely many non-equivalent formulas, and so φ can be expressed by choosing appropriate representatives. □

An application: number of individuals

Ehrenfeucht-Fraïssé games are a very useful tool to investigate the expressive power of a logic. For instance, consider the following question. In a given world w, the actual individuals are given by the equivalence classes $D/{\sim_w}$. Let us denote by $c(w)$ the number of these individuals, so that world w represents a state of affairs where there are exactly $c(w)$ individuals. Now, can $\texttt{InqBQ}$ express the question of how many individuals there actually are? In more formal terms, is there a sentence φ of $\texttt{InqBQ}$ with the following semantics?

$$M \vDash \varphi \iff \forall w, w' \in W : c(w) = c(w')$$

Using the game, we can show that a formula with this property does not exist, and this provides an excellent example of the sort of open questions that our result allows us to answer.

Proof sketch. For simplicity we consider the empty signature, but the proof easily generalizes to an arbitrary signature Σ. By contradiction: suppose a formula φ as above exists and define $n = \mathtt{Q}(\varphi)$. Toward a contradiction, we will present two models that entail the same formulas up to $\mathtt{Q}$-complexity n, but that disagree on the property $\forall w, w' \in W : c(w) = c(w')$.

For h, k positive natural numbers, define $\mathcal{M}_{\langle h,k\rangle}$ as the model with set of worlds $\{w_0, w_1\}$, with domain $[h] \times [k] = \{\langle a, b\rangle \,|1 \leq a \leq h, 1 \leq b \leq k\}$ and with $\sim$ defined by

$$\langle a, b\rangle \sim_{w_0} \langle a', b'\rangle \iff a = a' \qquad\qquad \langle a, b\rangle \sim_{w_1} \langle a', b'\rangle \iff b = b'$$

Clearly the property $c(w_0) = c(w_1)$ holds if and only if $h = k$. Moreover, given a sequence of elements $\overline{U} = \langle\langle a_1, b_1\rangle, \ldots, \langle a_p, b_p\rangle\rangle$, the set of atomic formulas with parameters in $\overline{U}$ supported at an info state is determined by the sets $E_1^{\overline{U}} = \{\langle i, j\rangle \,|a_i = a_j\}$ and $E_2^{\overline{U}} = \{\langle i, j\rangle \,|b_i = b_j\}$ and is independent from h and k.

Using the EF game we can show that $\mathcal{M}_{\langle n,n\rangle}$ and $\mathcal{M}_{\langle n,n+1\rangle}$ satisfy the same formulas of $\mathtt{Q}$ complexity up to n. To show this, we describe a winning strategy for **II** in the game $\mathrm{EF}_{m,n}(M_{\langle n,n\rangle}; M_{\langle n,n+1\rangle})$ for an arbitrary m. Suppose that the current position of the game is $\left\langle \mathcal{M}_{\langle n,n\rangle}, s, \overline{U}; \mathcal{M}_{\langle n,n+1\rangle}, s, \overline{V}\right\rangle$ (notice that the info states are the same). The strategy is determined by the following conditions:

If I plays an $\rightarrow$ move and chooses s' in one of the two models, then **II** chooses s' in the other. Notice that this condition ensures that the info states that appear in the current position are the same trough the game;

If I plays a $\forall$ or $\overline{\exists}$ move and chooses an element from the model $\mathcal{M}_{\langle n,n\rangle}$ obtaining the sequence $\overline{U'} = \overline{U}\,\langle a_l, b_l\rangle$, then **II** choose an element from the other model obtaining a sequence $\overline{V'} = \overline{V}\,\langle a'_l, b'_l\rangle$ such that $E_1^{\overline{U}} = E_1^{\overline{V}}$ and $E_2^{\overline{U}} = E_2^{\overline{V}}$. Notice that this can always be achieved as the sequences $\overline{U'}$ and $\overline{V'}$ have length at most n.
The case in which **I** chooses an element from the model $\mathcal{M}_{\langle n,n+1\rangle}$ is analogous.

At the end of the game, the equality $E_i^{\overline{U}} = E_i^{\overline{V}}$ for $\overline{U}$ and $\overline{V}$ the sequences of elements chosen in the two models ensures the winning condition for **II**, as wanted.

□

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