Properties of Local Homeomorphisms of Stone spaces and Priestley spaces

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Introduction. Exponentiability of objects and morphisms is one of the important good properties for a category. The problem of exponentiability has been studied in many contexts since 1940s. The reader interested in the subject is referred to: the articles [4], [1],[7], [8]; and the books [3], [5],[6].

Due to the existence of important dualities between Stone spaces and Boolean algebras, as well as between Priestley spaces and distributive lattices, our aim is to characterize exponentiable objects and exponentiable morphisms in the categories of Stone spaces and Priestley spaces. The presented work is part of the more extensive program, which aims to study local homeomorphisms of logical and partially ordered topological spaces which are important for logic e.g. Stone spaces, Priestley spaces, Spectral spaces, Esakia spaces. This is motivated by the importance of local homeomorhphisms not only in topology, but in algebraic geometry and other areas of mathematics due to their attractive properties.

Given objects X, Y in the small category $\mathbf{C}$ with finite limits, the object Y^X (if it exists in $\mathbf{C}$) is said to be an *exponential* of Y by X, if for any object A in $\mathbf{C}$ there is a natural bijection between the set of all morphisms from $A \times X$ to Y and the set of all morphisms from A to Y^X, i.e. $\mathbf{C}(A \times X, Y) \cong \mathbf{C}(A, Y^X)$. An object X of a category $\mathbf{C}$ is said to be *exponentiable* if the exponent Y^X exists in $\mathbf{C}$ for any object Y. Given object X in $\mathbf{C}$, consider the class $\mathbf{C}/X$ of morphisms $f : Y \to X$ with codomain X in $\mathbf{C}$. Let morphisms between members of the mentioned class be the obvious commutative triangles. It is easy to check that the family together with the defined morphisms between them is a category. It is the case that if $\mathbf{C}$ has all finite limits, then so does $\mathbf{C}/X$. Let us note that the product of two objects of $\mathbf{C}/X$ is a pullback in $\mathbf{C}$ with the obvious projection to X. As in the case of $\mathbf{C}$, given two morphisms $f : Y \to X$ and $g : Z \to X$ the object g^f (if it exists in $\mathbf{C}/X$) is said to be an exponential of g by f, if for any object $h : W \to X$ in $\mathbf{C}/X$ there is a natural bijection between the set of morphisms from $h \times_X f$ to g and the set of morphisms from h to g^f, i.e. $\mathbf{C}/X(h \times_X f, g) \cong \mathbf{C}/X(h, g^f)$. A morphism f of a category $\mathbf{C}$ is said to be *exponentiable morphism* if the exponent g^f exists in $\mathbf{C}/X$ for any morphism g of $\mathbf{C}$.

Note that for categories of sets with additional structure and structure preserving maps, the problem of exponentiability reduces to finding appropriate corresponding structure of the same kind on the set of structure-preserving maps. In the following subsections we state the main result already obtained regarding exponentiable objects and morphisms in the categories of Stone spaces and Priestley spaces. For brevity, the supporting lemmas and propositions are omitted.

We conclude the section with definition of the abovementioned notion of local homeomorphism between topological spaces. A map $f : X \to B$ between topological spaces X and B is said to be a *local homeomorphism* if each point x in X has an open neighborhood which is mapped homeomorphically by f onto an open subset of B. For more information about local homeomorphisms see [3], [6].

Local homeomorphisms as Exponentiable maps of Stone spaces. A compact, Hausdorff, and zero-dimensional topological space is called a *Stone space*. The first category we are interested in is the category of Stone spaces and continuous maps. Let us denote the mentioned category by **Stone**. Our investigation of exponentiability of objects in **Stone** showed that only the finite spaces are exponentiable (unlike the case of the category of all topological spaces where only core-compacts are exponentiable, that is the spaces where every neighborhood U of any point x has a sub-neighborhood V, such that every open cover of U contains a finite subcover of V; such spaces can be infinite [2],[4]). Note that by [1] compact Hausdorff topological space is exoponentiable iff it is finite. Below for Stone spaces and Priestley spaces we have similar results:

Proposition 1. *A Stone space X is exponentiable in* **Stone** *if and only if X is finite.*

After that we are able to prove the full characterization of exponentiable maps of Stone spaces. That is the following result holds:

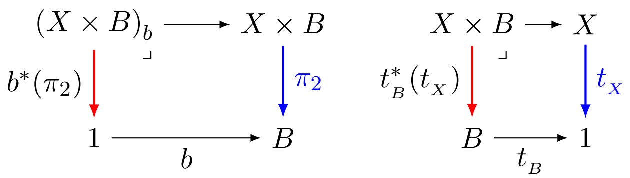
Proposition 2. *The map $f : X \to B$ between Stone spaces is exponentiable in* **Stone**$/B$ *if and only if f is a local homeomorphism.*

Exponentiability in Priestley spaces. A partially ordered topological space $(X, \leq)$ is called a Priestley space, if X is compact topological space and for any pair $x, y \in X$ with $x \nleq y$, there exists a clopen up-set U of X such that $x \in U$ and $y \notin U$. It turns out that the topology on a Priestley space is compact Hausdorff and zero-dimensional, i.e. is a Stone topology. The second category we are interested in is the category of Priestley spaces and continuous order-preserving maps. Let us denote this category by **PS** (**P**riestley **S**paces). Investigation of exponentiability of objects in **PS** showed that, similarly to the case of Stone spaces, only finite spaces are exponentiable in **PS**. Hence the following:

Proposition 3. *A Priestley space X is exponentiable in* **PS** *if and only if X is finite.*

Due to this fact, given a Priestley space B we get the following corollary about exponentiability of $\pi_2 : X \times B \to B$ in **PS**$/B$:

Corollary 3.1. *$\pi_2 : X \times B \to B$ is exponentiable in* **PS**$/B$ *if and only if X is finite.*



Moreover, we were able to prove a necessary condition for exponentiability of a map between Priestley spaces. An order preserving map $f : X \to B$ is called an interpolation-lifting map if given $x \leq y$ in X and $f(x) \leq b \leq f(y)$, there exists $x \leq z \leq y$ such that $f(z) = b$.

Proposition 4. *If $f : X \to B$ is exponentiable in* **PS**$/B$ *then f is interpolation-lifting.*

We are still unable to find a necessary and sufficient condition for exponentiability of Priestley maps. Already obtained results draw quite interesting picture of considered categories. Only the smallest part of the considered categories (only finite objects) have such strong property as exponentiability. Further work is in progress, namely we are investigating whether exponentiable morphisms in **PS** are precisely the local homeomorphisms that are also interpolation-lifting maps.

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