Recursive Enumerability Doesn't Always Give a Decidable Axiomatization

Stepan Kuznetsov*

Steklov Mathematical Institute, RAS (Moscow)

Valentina Lugovaya and Anastasiia Ryzhova

Moscow State University

It is well-known that if a theory (deductively closed set of formulae) over a well-behaved logic (for example, classical or intuitionistic logic) is recursively enumerable (r.e.), then it has a decidable, and even a primitively recursive axiomatization [2]. This observation, known as *Craig's theorem,* or Craig's trick, is indeed very general. If we denote the deductive closure (set of theorems) for an axiomatization $\mathcal{A}$ by $[\mathcal{A}]$ and let $[\mathcal{A}]$ be recursively enumerated as follows: $\varphi_1, \varphi_2, \varphi_3, \ldots$ ($\varphi_k = f(k)$, where f is a computable function), then the set $\mathcal{A}' = \{\varphi_1, \varphi_2 \wedge \varphi_2, \varphi_3 \wedge \varphi_3 \wedge \varphi_3, \ldots\}$ will be decidable (the decision algorithm, given a formula ψ, starts enumerating $\mathcal{A}'$, compares the elements with ψ, and stops with the answer "no" when the size of the formula exceeds the size of ψ: further formulae will be only bigger), and, on the other hand, $\mathcal{A}'$ serves as an alternative axiomatization for the theory, since $[\mathcal{A}'] = [\mathcal{A}]$.

The only thing we need from the logic for this construction to work is the following property: for any formula ψ there exists, and can be effectively constructed, an equivalent formula ψ' of greater size than ψ. Then we take $\mathcal{A}' = \{\varphi_1, \varphi_2', \varphi_3'', \ldots\}$ as the needed decidable axiomatization: since $'$ increases the size of formula, the n-th formula in this sequence has size at least n; therefore, in our search for a given ψ in $\mathcal{A}'$ we have to check only a

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finite number of formulae. This works even for substructural systems that don't enjoy $\psi \leftrightarrow \psi \wedge \psi$. For example, once there is an operation $\circ$ that has a unit $\mathbf{1}$, Craig's theorem is valid: $A \leftrightarrow A \circ \mathbf{1} = A'$.

Thus, it looks interesting to find a logic for which Craig's theorem fails. Of course, one could easily construct degenerate examples, like a "logic" without any rules of inference: then $[\mathcal{A}]$ is always $\mathcal{A}$, and if it was r.e., but not decidable, it doesn't have a decidable axiomatization. So we're seeking for an example among interesting, useful logical systems.

And such an example exists—it is the product-free fragment of the *Lambek calculus* [3]. We denote this calculus by **L** and present it here as a Gentzen-style sequential calculus; a non-sequential ("Hilbert-style") version also exists [4]. Formulae of **L** are built from a set of variables $\mathrm{Var} = \{p_0, p_1, p_2, p_3, \dots\}$ using two binary connectives, $\backslash$ and $/$. Sequents are expressions of the form $A_1, \dots, A_n \to B$, where A_i and B are formulae and $n \geq 1$ (empty antecedents are not allowed). The axioms and rules of **L** are as follows (here capital Greek letters denote sequences of formulae):

$$\overline{A \to A}$$

$$\frac{A, \Pi \to B}{\Pi \to A \backslash B} \text{ where } \Pi \text{ is non-empty} \qquad \frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, \Pi, A \backslash B, \Delta \to C}$$

$$\frac{\Pi, A \to B}{\Pi \to B \,/\, A} \text{ where } \Pi \text{ is non-empty} \qquad \frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, B \,/\, A, \Pi, \Delta \to C}$$

$$\frac{\Pi \to A \quad \Gamma, A, \Delta \to C}{\Gamma, \Pi, \Delta \to C} \text{ (cut)}$$

Let $\mathcal{A}$ be an arbitrary set of sequents. We say that a sequent $\Pi \to A$ is derivable from $\mathcal{A}$ (denoted by $\mathcal{A} \vdash_{\mathbf{L}} \Pi \to A$), if there exists a derivation tree where inner nodes are applications of rules (including cut: in this setting it is not eliminable), and leafs are instances of axioms or sequents from $\mathcal{A}$. The theory axiomatized by $\mathcal{A}$ (the deductive closure of $\mathcal{A}$) is $[\mathcal{A}] = \{\Pi \to A \mid \mathcal{A} \vdash_{\mathbf{L}} \Pi \to A\}$. Clearly, if $\mathcal{A}$ is r.e., then so is $[\mathcal{A}]$. Finally, $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent, $\mathcal{A}_1 \approx \mathcal{A}_2$, if $[\mathcal{A}_1] = [\mathcal{A}_2]$.

Theorem. *There exists such a recursively enumerable $\mathcal{A}$ that there is no decidable $\mathcal{A}'$ equivalent to $\mathcal{A}$.*

Let $q = p_0$ and let $\mathcal{E} = \{p_i \to q \mid i \geq 1\}$.

Lemma. *If $\mathcal{A} \subsetneq \mathcal{E}$ and $\mathcal{A}' \approx \mathcal{A}$, then $\mathcal{A}' \cap \mathcal{E} = \mathcal{A}$.*

This Lemma immediately yields our goal: if $\mathcal{A}$ is a recursively enumerable undecidable subset of $\mathcal{E}$, it gives undecidability of any $\mathcal{A}'$ equivalent to $\mathcal{A}$.

We prove the Lemma by a semantic argument, via formal language models for $\mathbf{L}$. Let Σ be an alphabet; Σ^+ stands for the set of all non-empty words over Σ. An interpretation w is a function that maps formulae of $\mathbf{L}$ to subsets of Σ^+, defined arbitrarily on variables and propagated as follows:

$$w(A \setminus B) = w(A) \setminus w(B) = \{u \in \Sigma^+ \mid (\forall v \in w(A))\, vu \in w(B)\}$$
$$w(B \,/\, A) = w(B) \,/\, w(A) = \{u \in \Sigma^+ \mid (\forall v \in w(A))\, uv \in w(B)\}$$

A sequent $A_1, \ldots, A_n \to B$ is true under interpretation w, if $w(A_1) \cdot \ldots \cdot w(A_n) \subseteq w(B)$, where $M \cdot N = \{uv \mid u \in M, v \in N\}$. The calculus is sound w.r.t. this interpretation: if all formulae of $\mathcal{A}$ are true under w and $\mathcal{A} \vdash_{\mathbf{L}} \Pi \to B$, then $\Pi \to B$ is also true under w. (A weak completeness result, for $\mathcal{A} = \varnothing$, is shown in [1]. Here we need only soundness.)

We consider a countable alphabet, $\Sigma = \{a_1, a_2, \ldots\}$.

First, we show that $\mathcal{A} \not\vdash_{\mathbf{L}} p_i \to p_j$ for $i \neq j$, $i, j \geq 1$. Consider an interpretation $w_1(p_i) = \{a_i\}$, $w_1(q) = \Sigma^+$. All sequents from $\mathcal{A}$ are true under w_1, while $p_i \to p_j$ isn't. Therefore, $(p_i \to p_j) \notin \mathcal{A}'$ if $i \neq j$, $i, j \geq 1$.

Second, we show that $\mathcal{A} \not\vdash_{\mathbf{L}} E_1 \setminus E_2 \to p_i$ and $\mathcal{A} \not\vdash_{\mathbf{L}} E_2 \,/\, E_1 \to p_i$ for any $i \geq 0$ and any formulae E_1 and E_2. The counter-interpretation here is as follows: $w_2(p_i) = \{a_i\} \cup \Sigma^{\geq 2}$, $w_2(q) = \{a_j \mid (p_j \to q) \in \mathcal{A}\} \cup \Sigma^{\geq 2}$, where $\Sigma^{\geq 2}$ is the set of all words of length at least 2. All sequents from $\mathcal{A}$ are true under w_2. By induction on A we show that $w_2(A) \supseteq \Sigma^{\geq 2}$ for any formula A. Then, since uv is always in $\Sigma^{\geq 2} \subseteq w_2(E_2)$, we have $w_2(E_1 \setminus E_2) = w_2(E_2 \,/\, E_1) = \Sigma^+$, but $w_2(p_i)$ is not Σ^+ for any i (including 0).

Third, we show that if $\mathcal{A} \vdash_{\mathbf{L}} p_i \to q$, then $(p_i \to q) \in \mathcal{A}$. If not, then interpretation w_2 defined above falsifies $p_i \to q$ keeping all sequents in $\mathcal{A}$ true. This yields $\mathcal{A}' \cap \mathcal{E} \subseteq \mathcal{A}$ (since all sequents in $\mathcal{A}'$ are derivable from $\mathcal{A}$).

Finally, we establish the converse inclusion by contraposition. Let $(p_k \to q) \notin \mathcal{A}'$ and show that $(p_k \to q) \notin \mathcal{A}$. Consider the following interpretation: $w_3(p_i) = \{a_i\} \cup \Sigma^{\geq 2}$, $w_3(q) = \{a_j \mid (p_j \to q) \in \mathcal{A}'\} \cup \Sigma^{\geq 2}$. Evidently, w_3 falsifies $p_k \to q$. It remains to show that all sequents from $\mathcal{A}'$ are true under w_3. There are several possible cases for a sequent from $\mathcal{A}'$.

1. The sequent is of the form $A \to A$ (including $q \to q$ or $p_i \to p_i$). This is an axiom, it is true everywhere.

2. The sequent is of the form $p_i \to q$. Then it is true by definition.

3. The sequent is of the form $p_i \to p_j$, $i \neq j$, $i, j \geq 1$. Then this sequent is not derivable from $\mathcal{A}$ (see above) and therefore cannot belong to $\mathcal{A}'$.

4. The sequent is of the form $E_1 \setminus E_2 \to p_i$ or $E_2 \,/\, E_2 \to p_i$. Again, it couldn't be derivable from $\mathcal{A}$ and couldn't belong to $\mathcal{A}'$.

5. The sequent is of the form $A \to F_1 \setminus F_2$ or $A \to F_2 \,/\, F_1$. As for w_2, for w_3 we have $w_3(F_1 \setminus F_2) = w_3(F_2 \,/\, F_1) = \Sigma^+$. The sequent is true.

Hence, $\mathcal{A}' \nvdash_{\mathbf{L}} p_k \to q$, therefore $(p_k \to q) \notin \mathcal{A}$. This finishes the proof.

Notice that this result is not at all robust: slight modifications of the calculus restore Craig's theorem. First, actually one can increase the size of all formulae, *except* variables, by the following equivalences: $A \,/\, B \leftrightarrow A \,/((A \,/\, B) \setminus A)$ and $B \setminus A \leftrightarrow (A \,/(B \setminus A)) \setminus A$. In our construction, we played on an infinite number of variables, for which such increasing is impossible. Thus, Craig's theorem holds for any fragment of **L** with a finite set of variables. Second, if we allow sequents with empty left-hand sides (and remove non-emptiness restrictions from the rules of **L**), we have $A \leftrightarrow (A \,/\, A) \setminus A$ for any formula A, which also yields Craig's theorem.

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