Johnstone-Gleason covers for partially ordered sets

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We are investigating precise relationship between the classical Gleason cover, its various subsequent generalizations for general topological spaces, and the Gleason cover of a topos by Johnstone, in case of not necessarily sober T_0 Alexandroff spaces, i. e. arbitrary partially ordered sets with Alexandroff topology.

The classical Gleason cover $\tilde{X}$ of a compact Hausdorff space X is the Stone dual of the complete Boolean algebra of its regular closed sets (regular open sets may be used, equivalently) [7]. It is equipped with an irreducible surjective map $\tilde{X} \to X$ and is defined uniquely up to homeomorphism.

Subsequently, many authors have introduced various generalizations of this construction to many classes of spaces — see, among others, [6, 8, 9, 4, 5, 1, 2, 12], under the name of *absolute*.

In 1980, Johnstone introduced a construction of the Gleason cover for arbitrary elementary topos [10, 11] which in particular gives certain version of absolute for any topological spaces and, more generally, of *locales* in point-free topology. Moreover, the Johnstone-Gleason cover $\tilde{\mathscr{E}}$ of a topos $\mathscr{E}$ can be uniquely characterized as the topos of sheaves over a minimal compact regular extremally disconnected locale in $\mathscr{E}$.

Johnstone has shown that his construction indeed gives what is commonly known as the Iliadis absolute for regular topological spaces. More precisely, he has shown that for the topos $\mathrm{Sh}(X)$ of sheaves on a regular space X his construction gives the topos equivalent to $\mathrm{Sh}(\tilde{X})$, where $\tilde{X}$ is the Iliadis absolute of X. However extending such equivalence to not necessarily regular spaces seems to be more subtle and, to our best knowledge, has not been investigated in the existing literature. This question becomes especially problematic for non-sober spaces, given that the sheaf topos construction (even already the passage from a space to the lattice of its open sets) does not distinguish between a space and its soberification. Although in [11] several results deal with non-sober spaces, it is not clear how is his construction related to any possible absolute constructions in the literature cited above.

In [10] also the explicit construction of his Gleason cover is provided for the topos of sheaves on a finite category. In the particular case when this category corresponds to a finite partially ordered set, we show that this construction is isomorphic to one obtained by us in [3]. However, in our work we encountered an interesting phenomenon not addressed to by Johnstone. When trying to prove uniqueness of the Gleason cover of a finite T_0 space, it turned out that we cannot use the key projectivity property of extremally disconnected spaces, which does not hold in the category of topological spaces and arbitrary continuous maps. For general spaces uniqueness of the Gleason cover is usually established in a different way — namely, by restricting the class of maps between spaces. In the papers cited above, either the class of *proper* maps or that of θ-*continuous* maps.

In [3], unaware of this fact, we acted differently to circumvent non-projectivity of extremally disconnected spaces with respect to arbitrary continuous maps. Namely, we observed that the Gleason cover map in this case is a so called *co-local homeomorphism*, which by definition means that it becomes a local homeomorphism if we replace spaces corresponding to

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finite posets with the ones corresponding to the same posets but with the opposite ordering. It turned out that the co-local homeomorphism property can be used to prove uniqueness of the Gleason cover.

We address the question of extending this approach to Alexandroff topologies of arbitrary partially ordered sets. When this topology is sober, the generalization is more or less straightforward. As is well known, the Alexandroff topology of a poset P is sober if and only if P is *Noetherian*, i. e. does not possess infinite ascending chains. In this case, the construction of the Gleason cover just repeats that of the finite case: we take disjoint union of downsets of all maximal elements of P. Note that the resulting map is easily seen to be a co-local homeomorphism, so we can generalize our approach to the proof of uniqueness to this case. However, for non-sober P, the Johnstone-Gleason cover will not necessarily produce a spatial locale, let alone an Alexandroff space of some other poset.

We will finish with a characterization of those posets P for which the Gleason cover of the corresponding Alexandroff space $\mathrm{A}(P)$ is itself Alexandroff. Namely, we will prove

Theorem. *For a poset P, there is a poset $\tilde{P}$ such that $\mathrm{A}(\tilde{P})$ is homeomorphic to the Gleason cover of $\mathrm{A}(P)$ if and only if P has the following property: for any $x \in P$ and any infinite set $Y \subseteq P$ with $x \leqslant y$ for all $y \in Y$ there exist $y, y' \in Y$ with $y \neq y'$ and $z \in P$ with $y \leqslant z$ and $y' \leqslant z$.*

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