Modal logics with a restricted universal modality

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Abstract

In this study we give syntax, semantics and axiomatization for a family of modal logics with restricted form of universal modality. The language of these family of modal logics allows the expression of Boolean combinations of formulas of the form $\exists\phi$ where ϕ is a formula of the ordinary language of modal logic and $\exists$ is the universal diamond. By means of a tableaux-based approach, we provide decision procedures for their satisfiability problem.

1 Syntax and Semantics

Our language $L_r(\Box, \exists)$ is defined using a countable set BV of Boolean variables (p, q, r, etc). We inductively define the set $t(BV)$ of terms (with typical members noted A, B, C, etc) as follows:

$$A ::= p \mid 1 \mid -A \mid (A \cap B) \mid \Box A.$$

We inductively define the set $f(BV)$ of formulas (with typical members noted ϕ, ψ, κ, etc) as follows:

$$\phi ::= \exists A \mid \top \mid \neg\phi \mid (\phi \wedge \psi).$$

A frame is an ordered pair $F = (W, R)$ where W is a non-empty set of possible worlds (x, y, z, etc) and R is a binary relation on W. For all $x \in W$, let $R(x)$ be the set of all $y \in W$ such that xRy. A valuation based on F is a function V assigning to each Boolean variable p a subset $V(p)$ of W. V induces a function $(\cdot)^V$ assigning to each term A a subset $(A)^V$ of W such that $(p)^V = V(p)$, $(1)^V = W$, $(-A)^V = W \setminus (A)^V$, $(A \cap B)^V = (A)^V \cap (B)^V$, and $(\Box A)^V = \{x : R(x) \subseteq (A)^V\}$.

A model is an ordered triple $M = (W, R, V)$ where $F = (W, R)$ is a frame and V is a valuation based on F. Let $\models$ be the satisfiability relation defined between models and formulas. The Boolean connectives are interpreted as usual. We define $M \models \exists A$ iff $(A)^V \neq \emptyset$.

Let F be a frame. A formula ϕ is said to be valid in F, in symbols $F \models \phi$, iff for all models M based on $F, M \models \phi$. Let C be a class of frames. A formula ϕ is said to be valid in C, in symbols $C \models \phi$, iff for all frames F in C, $F \models \phi$.

The syntax and semantics defined as above constitute a fragment of ordinary modal logic with the universal modality. In this fragment, one can easily translate the Contact Logics considered by Vakarelov (2007) and Balbiani *et al.* (2007)

2 Tableaux Approaches, Soundness and Completeness

The language is based on two types of expressions: terms and formulas. For these reasons, tableau nodes will be labeled by the following types of expressions: formulas, expressions of the form $x : A$ and expressions the form $x \triangle y$, where x, y are symbols and A is a Boolean term. Given a formula ϕ, its initial tableau is the labeled tree consisting of exactly one node labelled with ϕ. The rules presented in Annex B are applied in a standard way by extending branches of constructed trees.

Definition 2.1. *A branch is said to be closed if and only if one of the following conditions holds:*

(i) *it contains a node labeled with* $x : -1$*;*

(ii) *it contains two nodes respectively labeled with* $x : A$*,* $x : -A$*;*

(iii) *it contains a node labeled with* $\perp$*.*

In order to prove that the tableaux of satisfiable formulas cannot be closed, we introduce the concept of interpretability of a branch in a model.

Definition 2.2. *Let $M = (W, R, V)$ be a model. Let β be a branch in a tableau and W' be the set of all variables occurring in β. The branch β is said to be interpretable in M if there exists a function $f : W' \to W$ such that: if ϕ occurs in β, then $M \models \phi$, if $x \triangle y$ occurs in β, then $f(x)Rf(y)$ and if $x : A$ occurs in β, then $f(x) \in \bar{V}(A)$.*

Now, we show the soundness of the tableau rules for modal logics with a restricted universal modality.

Proposition 2.3. *Let $M = (W, R, V)$ be a model and ϕ be a formula. If $M \models \phi$, then every tableau computed from the initial tableau of ϕ is interpretable in M and is therefore open.*

The previous proposition shows that the tableau rules for modal logics with a restricted universal modality preserves property of interpretability in general models. As for the termination property, we need to define the following preliminary definitions.

Definition 2.4. *Let β be a branch of some tableau and x be a symbol in β. Let $(A_1, ..., A_n)$ be a list of all modal terms A such that $x : -\Box A$ is in β. We will say that x is successor-free in β if there exists $i \in \{1, .., n\}$ such that for all y in β, if $x \triangle y$ is in β then $y : -A_i$ is not β. Let $term(x, \beta) = \{a : \ x : a \text{ occurs in } \beta\}$. We will say that x is twin-free in β if for all y in β, if $terms(x, \beta) = terms(y, \beta)$ then for all $u_1, ..., u_n$ in β, if $t\Delta u_1$ is in $\beta, ..., t\Delta u_n$ is in β then either $u_1 : -A_1$ is not in $\beta, ...,$ or $u_n : -A_n$ is not in β.*

The strategy is the following: (i) Apply all rules (except the $-\Box$ rule) as much as possible, (ii) Apply the $-\Box$ rule to $x : -\Box A$ in β, when x is successor-free in β and twin-free in β and (iii) If the $-\Box$ rule has been applied in step *(ii)* then go to step *(i)* else halt.

Now, let us show how the termination strategy will terminate. Note that for any symbol occurring in branches of a tableau constructed from ϕ's initial tableau, $term(x;\beta)$ contains only sub-terms or complements of sub-terms of ϕ. There exists finitely many sub-terms of ϕ. Consequently, at some point of the computation, in each branch of the constructed tree, each successor-free symbol is not twin-free. Thus, the strategy terminates.

Definition 2.5. *Let t be a tableau obtained after applying our strategy as much as possible. Let β be a branch in t. Suppose β is open. Let M be the model defined as follows:*

(i) *W is the set of all non successor-free x in β,*

(ii) *R is the binary relation on W defined by xRy iff either $x\triangle y$ occurs in β or there exists a symbol z in β such that z is successor free in β, $term(y,\beta) = term(z,\beta)$ and $x\triangle z$ occurs in β.*

(iii) *$V(p)$ is the set of all $x \in W$ such that β contains the information $x : p$.*

The following lemma is crucial for proving the completeness of our method.

Lemma 2.6. *Let t be a tableau, β be a branch in t and $M = (W, R, V)$ be the model defined above. We have the following:*

(i) *If β contains $x : A$ and $x \in W$, then $x \in V(A)$,*

(ii) *If β contains a formula ϕ, then $M \models \phi$.*

Now, we are ready to present the completeness theorem

Theorem 2.7. *Let ϕ be a formula and t a tableau obtained from the initial tableau of $\neg\phi$ by applying the tableau rules and our strategy as much as possible. If ϕ is valid in the class of all models then t is closed.*

3 Variant

In this section, we extend our systems with adding new tableau rule for dense models. We give sound and complete tableaux based decision procedure for variant of our logic. Obviously, for symmetric and reflexive models, the tableaux approaches can be easily adapted as expected.

A model $\mathcal{M} = (W, R, V)$ is said to be dense if for all $x, y \in W$, if xRy then there exists $z \in W$ such that xRz and zRy. Let us consider the class of all dense models. In order to decide satisfiability with respect to this class, we should add the Den rule to our system. Obviously, Den is sound with respect to dense models, i.e. it preserves the interpretability property of tableaux.

Now, we have to define a strategy that will guarantee the soundness, the completeness and the termination of our tableaux-based system.

Let β be a branch and (x, y) be a pair of symbols occurring in β. We will say (x, y) is intermediate-free if $x\Delta y$ occurs in β and for all symbols z in β, either $x\Delta z$ does not occur in β, or $z\Delta y$ does not occur in β. Let β be a branch, x be a symbol occurring in β. We will say (x, y) is twin-free if $x\Delta y$ occurs in β and for all symbols z_1, z_2, z_3 occurring in β, if $term(x, \beta) = term(z_1, \beta)$ and $term(y, \beta) = term(z_3, \beta)$ then either $z_1\Delta z_2$ does not occur in β, or $z_1\Delta z_3$ does not occur in β, or $z_2\Delta z_3$ does not occur in β.

Our strategy is the following: (i) Apply the formula rules and the term rules as much as possible, (ii) Choose an intermediate-free twin-free pair (x, y) of symbols occurring in a branch β; Apply the rule (Den) to (x, y) and go to (i) otherwise go to (iii) and (iii) Halt.

Proposition 3.1. *Let $M = (W, R, V)$ be a model and ϕ be a formula. If $M \models \phi$, then every tableau computed from the initial tableau of ϕ is interpretable in M and is therefore open.*

Proposition 3.2. *Let ϕ be a formula. After a finite number of steps from the initial tableau of ϕ, no tableau rule can be applied.*

Lemma 3.3. *Let t be a tableau, β be a branch in t and M be a model for β. We have the following:*

(i) *If β contains $x : A$ and $x \in W$, then $x \in \bar{V}(A)$.*

(ii) *If β contains a formula ϕ, then $\mathcal{M} \models \phi$.*

Theorem 3.4. *Let ϕ be a formula and t a tableau obtained from the initial tableau of $\neg\phi$ by applying the tableau rules augmented with (Den) and following the above strategy. If ϕ is valid in the class of all dense models then t is closed.*

4 Conclusion

We have given sound and complete tableaux-based decision procedures for the satisfiability problem in our logic. The satisfiability problem with respect to the class of all dense models is decidable. Unfortunately, we do not know its exact computational complexity. Note that this decidability result is new; it does not seem that it can be easily obtained by means of an argument based on the filtration method, seeing that the filtration construction does not preserve the elementary property of density.

References

[1] Balbiani, P., Tinchev, T., Vakarelov, D.: *Modal logics for region-based theories of space.* Fundamenta Informaticæ **81** (2007) 29–82.

[2] Fitting, M.: *Tableau methods of proof for modal logics.* Notre Dame Journal of Formal Logic 13, 237-247, 1972.

[3] Vakarelov, D.: *Region-based theory of space: algebras of regions, representation theory, and logics.* In: *Mathematical Problems from Applied Logic. Logics for the XXIst Century. II.* Springer (2007) 267–348.

5 Annex A: Proofs

Proof of Proposition 2.3. Suppose $M \models \phi$. Since the initial tableau of ϕ consists of a single node labeled with ϕ, therefore the initial tableau of ϕ is interpretable in M. The fact that the tableau rules preserve the interpretability property in M follows from the strict similarity between the relational semantics and the tableau rules. $\Box$

Proof of Lemma 2.7. (i) The proof is done by induction on A. We consider the case $\Box A$, the other cases being left to the reader. Let $x \in W$. Thus, x is not successor free. Suppose $x : \Box A \in \beta$. We want to show that $x \in V(\Box A)$. Let $y \in W$ such that xRy. We have to show that $y : A$ is in β. We have to consider two cases. In the first case, $x\Delta y$ is in β. Since $x : \Box A$ is in β, therefore by the Box rule, $y : A$ is in β. In the second case, let z in β be such that z is successor-free in β, $term(y, \beta) = term(z, \beta)$ and $x\Delta z$ in β. Since $x : \Box a$ is in β, therefore by the Box rule we have that $z : A$ is in β. Since $term(y, \beta) = term(z, \beta)$, therefore $y : A$ is in β.

(ii) The proof is by done induction on ϕ. We consider the case $\exists A$, the other cases being left to the reader. Suppose β contains a formula $\exists A$. The rule $\exists$ is applied, $\exists y \in \beta$ such that $x : A$ occurs in β. By item (i), we have $x \in V(A)$. Therefore $M \models \exists A$. $\Box$

Proof of Theorem 2.8. Suppose t is open. Thus, t contains an open branch β. Let $M = (W, R, V)$ be the model for β. By the truth lemma, we have $M \models \neg\phi$, contradicting the validity of ϕ. $\Box$

Proof of Proposition 3.1. Let us prove the soundness and the completeness of our tableau system extended with (Den) and following the above strategy. Obviously, every tableau constructed, by following the above strategy, from the initial tableau of a formula ϕ satisfiable in a dense model will be open. Contraversely, suppose β is an open branch obtained, by means of our strategy, at the end of the tableau computation from an initial formula ϕ. Let W be the set of all x, y, etc occurring in β. As expected, we define on W the valuation V such that for all Boolean variables p, $V(p) = \{x \in W\colon x : p$ occurs in $\beta\}$. Now, for the accessibility relation R on W, it is defined as follows: for all $x, y \in W$, xRy iff $x\Delta y$ occurs in β. $\Box$

Proof of Proposition 3.2. In order to show that our strategy terminate, it suffices to follow an argument similar to the one developed in the previous section. Let us be more precise. Firstly, remark that in any branch β of a tableau constructed from the initial formula ϕ and for any x occurring in β, $term(x, \beta)$ only contains sub-terms or negation of sub-terms from ϕ. Seeing that there exists finitely many sub-terms from ϕ, at some point of the computation, each intermediate-free pair (x, y) of symbols occurring in a branch β is not twin-free. Therefore, our strategy terminates. $\Box$

Proof of Theorem 3.3. Let ϕ be a formula and t be a tableau obtained from the initial tableau of $\neg\phi$ by means of the tableau rules augmented with (Den). If ϕ is valid in the class of all dense models then t is closed. $\Box$

6 Annex B: Tableaux Rules

Conjunction Rule

$$\frac{\phi \wedge \psi}{\begin{array}{c}\phi \\ \psi\end{array}}$$

Disjunction Rule

$$\frac{\neg(\phi \wedge \psi)}{\neg\phi \;\big|\; \neg\psi}$$

Negation Rule

$$\frac{\neg\neg\phi}{\phi}$$

$\exists$ Rule

$$\frac{\exists A}{x : A}$$

$\neg\exists A$ Rule

$$\frac{\neg\exists A}{x : -A}$$

Box Rule

$$\frac{\begin{array}{c}x : \Box A \\ x \triangle y\end{array}}{y : A}$$

Negation Box Rule

$$\frac{x : -\Box A}{\begin{array}{c}x \triangle y \\ y : -A\end{array}}$$

Intersection Rule

$$\frac{x : A \cap B}{\begin{array}{c}x : A \\ x : B\end{array}}$$

Union Rule

$$\frac{x : -(A \cap B)}{x : -A \;\big|\; x : -B}$$

Negation Rule

$$\frac{x : - - A}{x : A}$$

(Den)

$$\frac{x \triangle y}{\begin{array}{c}x \triangle z \\ z \triangle y\end{array}}$$

In the $\exists$ Rule, x is a new symbol. In the $-\Box$ Rule, y is a new symbol. In the Den Rule, z is new symbol.