Subintuitionistic Logics with Linear Models

Dick de Jongh[1] and Fatemeh Shirmohammadzadeh Maleki[2]

[1] Institute for Logic, Language and Computation, University of Amsterdam, The Netherlands
D.H.J.deJongh@uva.nl
[2] School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran
f.shmaleki2012@yahoo.com

In the present research in progress we study logics with linear models originating from logics weaker than IPC. Logics weaker than IPC are divided into two kinds. One has the subintuitionistic logics with Kripke models studied by [4, 6]. Those have a basic logic F, Corsi's Logic. And one has weaker ones with neighborhood models described in [7, 12]. Their basic logic is WF. Linear extensions of Visser's Basic Propositional Logic, BPC, a relatively strong extension of F, have already been studied by [1, 2, 14]. We mostly consider the weaker logics with neighborhood models, but have an interest in logics with Kripke models as well. For us linear will mean transitivity plus connectedness (i.e. in the case of Kripke models, $\forall xyz\,((xRy \wedge xRz) \to (y \neq z \to yRz \vee zRy)))$.

The well-known Gödel-Dummett logic LC [9] is an extension of intuitionistic logic IPC with linear Kripke-models. Its best-known axiomatization over IPC is by $\mathcal{L}_1$, $(A \to B) \vee (B \to A)$.

Corsi's logic F and extensions. The logic F is axiomatized by

1. $A \to A \vee B$	*7.* $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
2. $B \to A \vee B$	*8.* $(A \to B) \wedge (B \to C) \to (A \to C)$
3. $A \wedge B \to A$	*9.* $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
4. $A \wedge B \to B$	*10.* $A \to A$
5. $\frac{A \quad B}{A \wedge B}$	*11.* $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$
6. $\frac{A \quad A \to B}{B}$	*12.* $\frac{A}{B \to A}$

The axioms 8, 9 and 11 are more descriptively named I, C and D. Corsi [4] proved completeness for Kripke models with an arbitrary relation R without stipulation of persistence of truth. One obtains IPC from F by adding

R: $A \wedge (A \to B) \to B$ (defines and is complete for reflexive Kripke frames)
T: $(A \to B) \to ((B \to C) \to (A \to C))$ (defines and is complete for transitive Kripke frames)
P: $p \to (\top \to p)$ (defines and is complete for persistent Kripke models).

Visser [14] already proved that over BPC=FTP[1], $\mathcal{L}_1$ is not complete with regard to linear models, but $\mathcal{L}_2$, $(A \to B) \vee ((A \to B) \to A)$ is, see also [2]. We consider besides $\mathcal{L}_1$, $\mathcal{L}_2$ even more connectedness schemes, which are equivalent over IPC but often not over weaker systems:

$(\mathcal{L}_3)$ $(A \to B) \vee ((A \to B) \to B)$
$(\mathcal{L}_4)$ $(A \to B \vee C) \to (A \to B) \vee (A \to C)$
$(\mathcal{L}_5)$ $(A \wedge B \to C) \to (A \to C) \vee (B \to C)$
$(\mathcal{L}_6)$ $((A \to B) \to B) \wedge ((B \to A) \to A) \to A \vee B$.

[1] Letter combinations like FTP will always mean that the schemes T and P are added to F.

One can even multiply these by applying rules admissible in IPC, $\vdash A \vee B$ iff $\vdash (A \to C) \wedge (B \to C) \to C$ (DR), and, more generally, $\vdash D \to A \vee B$ iff $\vdash D \wedge (A \to C) \wedge (B \to C) \to C$ (EDR).

We study these variations and obtain for example that $\mathcal{L}_1$, $\mathcal{L}_4$ and $\mathcal{L}_5$ are equivalent over F. Moreover, we show that $\mathcal{L}_1$ plus $\mathcal{L}_3$ prove $\mathcal{L}_2$ in F, so $\mathcal{L}_1$ plus $\mathcal{L}_3$ is complete for linear models over BPC. Presently we are extending our investigations to the slightly weaker logic FT which lacks persistence of truth.We also obtain modal companions for a number of the logics.

Neighborhood models and extensions of the logics WF and $\mathsf{WF_N}$. The logic WF can be obtained by deleting the axioms C, D and I from F, and replacing them by the corresponding rules like concluding $A \to B \wedge C$ from $A \to B$ and $A \to C$ (see [12]). Neighborhood frames describing the natural basic system WF were obtained in [12]. The NB-neighborhoods consist of pairs (X, Y) with the X and Y corresponding to the antecedent and consequent of implications.

Definition 1. *$\mathfrak{F} = \langle W, NB\rangle$ is called an **NB-frame** of subintuitionistic logic if $W \neq \emptyset$ and $NB\colon W \to \mathcal{P}((\mathcal{P}(W))^2)$ is such that: $\forall w \in W \quad (X \subseteq Y \;\Rightarrow\; (X, Y) \in NB(w))$. If $\mathfrak{M}$ is a model on $\mathfrak{F}$, $\mathfrak{M}, w \Vdash A \to B$ iff $(V(A), V(B)) \in NB(w)$.*

In N-neighborhood frames (also in [12]), closer to the neighborhood frames of modal logic, $\overline{X} \cup Y$ corresponds to implications instead of (X, Y). An additional rule N [5, 7] axiomatizes them over WF:

$$\frac{A \to B \vee C \quad C \to A \vee D \quad A \wedge C \wedge D \to B \quad A \wedge C \wedge B \to D}{(A \to B) \leftrightarrow (C \to D)} \qquad (\mathsf{N})$$

WF plus N is denoted as $\mathsf{WF_N}$. For extensions of $\mathsf{WF_N}$ modal companions can often be found.

Again linearity will be the combination of connectedness and transitivity of the neighborhood frames. But, connectedness as well as transitivity now concerns sets of worlds, not individual worlds. To make this more perspicuous we write $X \leq_w Y$ for $(X, Y) \in NB(w)$ (or for $\overline{X} \cup Y \in N(w)$ in the case of N-frames). Then we can call the frames transitive if, for all w and all $X \leq_w Y,\; Y \leq_w Z$ we have $X \leq_w Z$ as well. The formula I defines this property and is complete for the transitive NB-frames. For the N-frames it is similar. Note also that, since by definition $(X, X) \in NB(w)$ for each X, $\leq_w$ will always be reflexive. The straightforward,

for all $X, Y \in \mathcal{P}(W)$ and $w \in W$, $X \leq_w Y$ or $Y \leq_w X$,

will be called connected$_1$ and is defined by $\mathcal{L}_1$. This formula defines a similar property in N-frames, and is complete for both types of frames. We can see linearity$_1$ as the combination of connectedness$_1$ and transitivity of the neighborhood frames. The other $\mathcal{L}_i$ define more complicated connectedness properties and in that way lead to different linearity properties.

We can refine the results on F by discussing in which extensions of WF they are provable.

Proposition 1. *The systems $\mathsf{WFI}\mathcal{L}_1$, $\mathsf{WFI}\mathcal{L}_4$ and $\mathsf{WFI}\mathcal{L}_5$ are equivalent.*

Proposition 2. $\mathsf{WFI}\mathcal{L}_1\mathcal{L}_3$ *proves* $\mathsf{I}\mathcal{L}_2$. *The opposite direction is open.*

We may even add axioms that hitherto only played a role above F in Kripke models for that purpose. For example, the axiom R which defines reflexivity in Kripke models defines a property of quasi-reflexivity in neighborhood models.

Definition 1.
*$\mathfrak{F}$ is **quasi-reflexive** iff for all $w \in W$, if $(X, Y) \in NB(w)$ and $w \in X$, then $w \in Y$.*

Proposition 3. $\mathsf{WF_N}\mathsf{IR}\mathcal{L}_1 \Vdash \mathcal{L}_2$.

The finite models of $\mathsf{WFI}\mathcal{L}_1$ can be seen as collections of linear orderings on $\mathcal{P}(W)$ without any connection between the different $\leq_u$ and $\leq_v$. We bring more coherence by adding the axioms P and T. Note that the transitivity axiom I of F, is distinct from the transitivity axiom T of IPC. We then get linear models in which the role of conjunction and disjunction is different than in Kripke models as the following proposition exhibits.

Proposition 4. $\mathsf{WFIPT}\mathcal{L}_1$ *does not prove* C *and does not prove* D.

Modal companions. We consider the translation $\Box$ from L, the language of propositional logic, to $L_{\Box}$, the language of modal propositional logic. It is given by:

$p^{\Box} = p$;

$(A \wedge B)^{\Box} = A^{\Box} \wedge B^{\Box}$;

$(A \vee B)^{\Box} = A^{\Box} \vee B^{\Box}$;

$(A \to B)^{\Box} = \Box(A^{\Box} \to B^{\Box})$.

This translation was discussed independently by [4] and [8] for subintuitionistic logics with Kripke models. We investigated it for extensions of $\mathsf{WF_N}$ in [5, 7]. For example a modal companion EN of $\mathsf{WF_N}$ is obtained by adding necessitation to classical modal logic E. Here we get modal companions for all of the extensions of $\mathsf{WF_N}$ that we discuss. These modal logics will have linearity properties as well. For example a modal companion of the logic $\mathsf{WF_N}\mathcal{L}_1$ is ENL_1 axiomatized over EN by $\mathsf{L}_1 : \Box(A \to B) \vee \Box(B \to A)$.

References

[1] M. Alizadeh, M. Ardeshir. On the linear Lindenbaum algebra of basic propositional logic, Mathematical Logic Quarterly 50, 65-70, 2004.

[2] M. Alizadeh, M. Ardeshir, Amalgamation Property for the class of basic algebras and some of its natural subclasses, Arch. Math. Logic, 45(8): 913-930, 2006.

[3] B. Chellas, Modal logic: An Introduction, Cambridge University Press, 1980.

[4] G. Corsi, Weak Logics with strict implication, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 33:389-406, 1987.

[5] D. de Jongh, F. Shirmohammadzadeh Maleki, Subintuitionistic Logics and the Implications they Prove, Indagationes Mathematicae, 10.1016/j.indag.2018.01.013.

[6] D. de Jongh, F. Shirmohammadzadeh Maleki, Subintuitionistic Logics with Kripke Semantics, In 11th International Tbilisi Symposium on Logic, Language, and Computation, TbiLLC 2015, LNCS, pp 333-354, Volume 10148, Springer 2017.

[7] D. de Jongh, F. Shirmohammadzadeh Maleki, Two Neighborhood Semantics for Subintuitionistic Logic, To be published in the Proceedings of the 12th International Tbilisi Symposium on Logic, Language, and Computation, TbiLLC 2017.

[8] K. Došen, Modal Translations in K and D, in: M. de Rijke, ed., Diamonds and Defaults, Kluwer Academic Publishers, 103127, 1993.

[9] M. Dummett, A propositional calculus with denumerable matrix, Journal of Symbolic Logic, vol. 24, 97-106, 1959.

[10] P. Hájek, Metamathematics of Fuzzy Logic. Dordrecht: Kluwer, 1998.

[11] M. Moniri, F. Shirmohammadzadeh Maleki, Neighborhood Semantics for Basic and Intuitionistic Logic, Logic and Logical Philosophy, pp 339-355, Volume 24, 2015.

[12] F. Shirmohammadzadeh Maleki, D. de Jongh, Weak Subintuitionistic Logics, Logic Journal of the IGPL, 25 (2): 214-231, 2017.

[13] A. S. Troelstra, D. van Dalen, Constructivism in Mathematics, Vol. 1, North-Holland, 1988.

[14] A. Visser, A propositional logic with explicit fixed points, Studia Logica, 40:155-175, 1981.