An algebraic study of Łukasiewicz logic with hedges

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1 Introduction

In pragmatics, *hedges* are linguistic terms used to alter the impact of an utterance (see e.g. [7]). For an updated account of their study in the setting of fuzzy logic we refer the reader to [6], we shortly recall here some salient facts, for sake of completeness. The study of hedges in fuzzy logic can be dated back to Lakoff [11], although they were initially investigated with respect to their ability to make things fuzzier or less fuzzy [12]. The term hedge in fuzzy logic is currently defined [1] as "a particle, word or phrase that modifies the degree of membership of a predicate or a noun phrase in a set". Typical examples are adjectives as in:

This khinkali is *incredibly* good (stresser),
This khinkali is *modestly* good (depresser),

or adverbs as in:

Georgian hospitality is *definitely* warm (stresser),
Georgian hospitality is *slightly* warm (depresser).

Other clauses acting as hedges are those which directly refer to the truth of some sentence like: *it is quite true that*, *it is more or less true that*, etc. Any sentence with a hedge can be translated into one using clauses of the latter kind. In this formulation, they have been represented in fuzzy logic as functions from the set of truth values (typically the real unit interval) into itself that modify the meaning of a proposition by changing the membership function of the fuzzy set underlying the proposition (see [14]).

In the setting of mathematical fuzzy logic, Hájek proposes [8, 9, 10] to understand them as truth functions of new unary connectives, a kind of modality or truth modifiers. The mathematical interpretation of a truth-depressing (resp. stressing) hedge on a chain C of truth-values is an operator f such that for any $x \in C$:

$$f(x) \leq x \;\; (f(x) \geq x, \text{respectivelly}), \tag{1}$$

We present an algebraic study of the logic called $\mathcal{R}Ł$, which is an expansion of Łukasiewicz logic by a family of unary operators f_r that can be interpreted as *linguistic hedges*. Indeed, in the standard algebra of truth values $[0,1]$, the family of unary operators f_r will be interpreted as *multiplications by a scalar* in $[0,1]$. By the algebraic properties of $[0,1]$, these operators are truth-depressing, however, since Łukasiewicz has an involutive negation *, one also obtains a family of truth-stressing operators by considering the definable terms $(f_r(x^*))^*$. The expressive power of this logic is particularly interesting as it exactly amounts to the piecewise linear functions on the unital hypercube $[0,1]^I$. Furthermore, the equivalent algebraic semantics of this logic, called RMV-algebras, has a profound mathematical relevance as it forms a category which is equivalent to the one of Riesz spaces (=real vector lattices) with order unit.

2 Preliminaries

We briefly recall the definition of MV-algebras, the standard references are [2, 13].

Definition 2.1. An *MV-algebra* is a structure $(A, \oplus, ^*, 0)$ such that

$$(A, \oplus, 0) \text{ is a commutative monoid}, \qquad (x^*)^* = x,$$
$$x \oplus 0^* = 0^*, \qquad (x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x.$$

It is customary to define, in the language of MV-algebras, the following derived operations

$$1 \coloneqq 0^*, \; x \ominus y \coloneqq (x^* \oplus y)^* \text{ and } \mathtt{d}(x,y) \coloneqq (x \ominus y) \oplus (x \ominus y).$$

Example 2.2. The standard MV-algebra is given by the set $[0,1]$, where the operations are interpreted as $x \oplus y = \min(x+y,1)$ and $x^* = 1-x$. An easy calculation shows that in this algebra

$$x \ominus y = \max\{0, x-y\}, \text{ and } \mathtt{d}(x,y) = |x-y|.$$

We now give the definition of Riesz MV-algebras (RMV-algebras, for short), they are MV-algebras endowed with a scalar multiplication by elements of the real interval $[0,1]$. RMV-algebras were introduced in [5] and further studied in [4].

Definition 2.3. An *RMV-algebra* is an MV-algebra A endowed with an external multiplication f_r for every real number r in $[0,1]$, satisfying the following conditions. For every $x,y \in A$ and every $r,s \in [0,1]$,

$$f_r(x \ominus y) = f_r(x) \ominus f_r(y). \tag{RMV 1}$$

$$f_{r \ominus s}(x) = f_r(x) \ominus f_s(x). \tag{RMV 2}$$

$$f_r(f_s(x)) = f_{r \cdot s}(x). \tag{RMV 3}$$

$$f_1(x) = x. \tag{RMV 4}$$

where $r \cdot s$ indicates the product in $[0,1]$ and $r \ominus s \coloneqq \max\{0, r-s\}$.

A routine argument shows that the operators f_r satisfy (1) and indeed RMV-algebras form the equivalent algebraic semantics of Łukasiewicz logic expanded with those operators.

Definition 2.4.

- (RMV-ideals) An *ideal* of an RMV-algebra is a non-empty downset, closed under $\oplus$ and f_r, for all $r \in [0,1]$.
- (local RMV-algebra) An RMV-algebra A is called *local* if it contains a unique maximal ideal.
- (Radical of an RMV-algebra) The intersection of all maximal ideals of an RMV-algebra A is called the *radical* of A and denoted by $\mathtt{Rad}(A)$; we call *infinitesimals* the elements of $\mathtt{Rad}(A)$.
- (Quasi-constant functions) A function f from a set X into some ultrapower $[0,1]^*$ is called *quasi-constant* if for all $x,y \in X$, $\mathtt{d}(f(x),f(y)) \in \mathtt{Rad}([0,1]^*)$.
- (Order unit) If R is a Riesz space, an element $u \in R$ is called an *order unit* if for every $x \in R$ there exists $n \in \mathbb{N}$ such that $u + \overset{n \text{ times}}{\dots} + u \geq x$

It is well known that an order unit in a Riesz space allows to define the sup-norm on that space, however it is readily seen that the concept of order unit is not first-order definable. As a matter of fact, Theorem 3.1 below shows that this is only a limit of the language used to describe these structures, and a more clever choice allows everything to become equationally definable.

3 Main results

The mathematical import of RMV-algebras is illustrated by the following result.

Theorem 3.1 ([5])**.** *There is functor Γ_R from the category of Riesz spaces with order unit (and unit preserving Riesz-morphisms) to the category of RMV-algebras which is full, faithful and dense, hence it witnesses an equivalence of the aforementioned categories.*

Theorem 3.2. *Every RMV-algebra A has a unique RMV-subalgebra isomorphic to* $[0,1]$*. Hence, no proper sub-quasi-variety (* a fortiori*, no proper subvariety) of RMV-algebra exists.*

General results by Davey [3, Lemma 2.1, Corollary 2.2] allow to prove a sheaf representation for RMV-algebras (hence also for Riesz spaces with unit).

Theorem 3.3. *Every RMV-algebra is isomorphic to the algebra of global sections of a sheaf of local MV-algebras over a compact Hausdorff space.*

Motivated by the previous result, we study the class of local RMV-algebras. Our results are abridged by the following characterisation theorem.

Theorem 3.4. *Let A be an RMV-algebra. The following are equivalent:*

1. *A is local,*
2. *A is generated by its radical,*
3. *$A/\mathtt{Rad}(A) \cong [0,1]$,*
4. *A is isomorphic to an algebra of quasi-constant functions,*
5. *A is isomorphic to $\Gamma_R(\mathbb{R} \overrightarrow{\times} W, (1,0))$, for some Riesz space W, were $\overrightarrow{\times}$ indicates the lexicographic product.*

We conclude with a somehow surprising result, when compared to Theorem 3.1. Indeed, the proper subclass of local RMV-algebras is equivalent to the full class of Riesz spaces (regardless whether they admit an order unit).

Theorem 3.5. *There is a categorical equivalence between Riesz spaces and local RMV-algebras.*

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