Analysis of the multi-modal logics for modal maps*

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Plain maps. Initially, we consider maps $f : X \to Y$ between two sets. Suppose that X and Y are disjoint sets. consider a Kripke frame $\mathfrak{F}_f = (W_f, R_f)$, where $W_f = X \sqcup Y$, $R_f = f$, i.e. we say that, pair of points $(x, y) \in W_f \times W_f$ is in the relation R_f, iff $f(x) = y$. The resulting Kripke frames are called Functional Frames. We say that the height of a frame $\mathfrak{F} = (W, R)$ is 2 if there exists $w, u \in W$, such that uRw and for any triple of distinct points $(u, v, w) \in W \times W \times W$ either uRv or vRw fails. We say that a Kripke frame $\mathfrak{F} = (W, R)$ has no branching, if for any triple of points $(u, v, w) \in W \times W \times W$ either uRv or uRw fails. Irreflexive frames of height ≤ 2 are characterized by a formula $\Box\Box\bot$, the no branching property is characterized by a formula $\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q)$. We show that a Kripke frame is a Functional Frame iff it is irreflexsive, non branching frame of height ≤ 2. The mentioned two formulas define the class of Functional Frames. Denote

$$K_f = K + \big(\Box\Box\bot\big) + \big(\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q)\big)$$

Proposition 1. *The modal logic K_f is sound and complete with respect to the class of Functional Frames.*

Proposition 2. *K_f has the finite model property.*

We show that although the class of Functional Frames is modally definable, the subclasses of injective and surjective functional frames are not. If we extend the modal language by using four temporal operators $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\overrightarrow{\Diamond}$ and $\overleftarrow{\Diamond}$, then the injective and surjective functional frames become definable. We interpret temporal operators as follows for a Kripke frame $\mathfrak{F} = (W, R)$ and $w \in W$,

1. $w \models \overrightarrow{\Box} p$ iff $\forall u \in W$, wRu implies $u \models p$.
2. $w \models \overleftarrow{\Box} p$ iff $\forall u \in W$, uRw implies $u \models p$.
3. $\overrightarrow{\Diamond} p = \neg \overrightarrow{\Box} \neg p$, $\overleftarrow{\Diamond} p = \neg \overleftarrow{\Box} \neg p$

We show that in the temporal language injective Functional Frames are determined by the formula

$$p \to \overrightarrow{\Box}\overleftarrow{\Box} p,$$

while surjective Function Frames are determined by the formula

$$\overrightarrow{\Diamond} \top \vee \overleftarrow{\Diamond} \top.$$

Order preserving maps. We consider the maps $f : \mathfrak{F}_1 \to \mathfrak{F}_2$ between Kripke frames $\mathfrak{F}_1 = (W_1, R_1)$ and $\mathfrak{F}_2 = (W_2, R_2)$. The *Relational Functional Frame* (RFF for brevity) associated with f is a bi-relational frame $f_R = (W, R, R_f)$, where $W = W_1 \sqcup W_2$, $R = R_1 \sqcup R_2$ and $R_f = f$. We say xRy if either xR_1y or xR_2y.

Note that (W, R_f) is a functional frame. In addition, the Relational Functional Frame f_R possesses the following *coherence property*: for any points $x, y \in W$, if $R_f(x) \neq \emptyset$ and $xRy \vee yRx$, then $R_f(y) \neq \emptyset$.

Proposition 3. *A bi-relational Kripke frame $\mathfrak{F} = (W, R, R_f)$ is an RFF (Relational Functional Frame) if and only if R_f is irreflexive, its height is less than 3, it is non-branching and R_f, R have the coherence property.*

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Since we deal with bi-relational frames, in the syntactic signature we now have $\Box$, $\Diamond$ for R_f and $\boxdot$, $\Diamonddot$ for R, hence we switch to bi-modal language. Here $\Box$, $\Diamond$ are interpreted as in Functional Frames and the $\boxdot$, $\Diamonddot$ are interpreted as follows

- For any formula φ in a language, $\boxdot\varphi$ is satisfiable in $w \in W$ if φ is true in all R-successors of w.
- For any formula φ in a language, $\Diamonddot\varphi$ is satisfiable in $w \in W$ if there exists an R-successor $u \in W$ of w, such that φ is satisfiable in u.

The Coherence Property in RFFs corresponds to the following formulas

$$\Diamond\top \to \boxdot\Diamond\top$$
$$\Diamonddot\Diamond\top \to \Diamond\top$$

We show that the class of all RFFs is modally definable in the bi-modal language. Let K_R be defined as follows

$$K_R = K_{\Box\boxdot} + \big(\Box\Box\bot\big) + \big(\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q)\big) + \big(\Diamond\top \to \boxdot\Diamond\top\big) + \big(\Diamonddot\Diamond\top \to \Diamond\top\big)$$

Proposition 4. *Bi-modal logic K_R is sound and complete with respect to the class of all Relational Functional Frames.*

We now impose some natural conditions on the underlying map f to see if the class of resulting RFFs is modally definable and finitely axiomatizable. Recall, that a map $f : \mathfrak{F}_1 \to \mathfrak{F}_2$ is *preserving* iff wR_1v implies $f(w)R_2f(v)$ for all $w, v \in W_1$. The map f is called *reflecting* iff for any $w \in W_1$ and $u \in W_2$, whenever $f(w)R_2u$ holds, there exists $u' \in W_1$ such that wR_1u' and $f(u') = u$. The map f is called a *p-morphism* iff it is both preserving and reflecting.

Let us adopt the following notation for the corresponding logics: Let K_{pre} denote the bi-modal logic of all RFFs with f preserving; let K_{ref} denote the logic of all RFFs with f reflecting and let K_p denote the logic of all RFFs with f a p-morphism.

Proposition 5. *The three logics are axiomatized as follows:*

$$K_{pre} = K_R + \big(\Diamonddot\Diamond p \to \Diamond\Diamonddot p\big)$$
$$K_{ref} = K_R + \big(\Diamond\Diamonddot p \to \Diamonddot\Diamond p\big)$$
$$K_p = K_R + \big(\Diamonddot\Diamond p \leftrightarrow \Diamond\Diamonddot p\big)$$

and define their respective classes of RFFs.

Moreover, we can show that K_R and K_{pre} logics are determined by their finite frames, i.e.:

Proposition 6. *K_R and K_{pre} have the finite model property.*

From the fmp and the finite axiomatization we conclude that K_R and K_{pre} logics are decidable and in fact, their decision problem turns out to be in PSPACE.

Continuous maps. Now instead of relations, we equip the domain and co-domain of a Functional Frame with topological structure. Suppose $f : X_1 \to X_2$ is a map between topological spaces (X_1, τ_1) and (X_2, τ_2). Let us introduce $f_\tau = (X, \tau, R_f)$ topological structure, where $X = X_1 \sqcup X_2$, τ is a topology generated by $\tau_1 \sqcup \tau_2$, and $R_f = f$. A topological structure $f_\tau = (X, \tau, R_f)$ is called a *Topological Functional Frame* (TFF for brevity) if (X, R_f) is a Functional Frame and $R_f^{-1}(X) = \{x \in X \mid \exists y \in X \text{ with } yR_fx\}$ is clopen (simultaneously closed and open according to τ). The latter condition is the Coherence Property for TFFs.

Again, due to existence of two, topological and function structures, we have two kinds of modal operators in our language, $\Box$, $\Diamond$ and $\boxdot$, $\Diamonddot$ respectively. The operator $\Diamond$ is interpreted as f^{-1}. The operator $\boxdot$ is interpreted as topological *Interior* operator and the operator $\Diamonddot$ – as topological *Closure* operator.

We show that the class of all TFFs (Topological Functional Frames) is modally definable. The bi-modal logic of Topological Functional Frames is denoted by $S4_R$. We axiomatize this logic as follows:

$$S4_R = K_R + (\boxdot p \to p) + (\boxdot p \to \boxdot\boxdot p).$$

Proposition 7. *Bi-modal logic $S4_R$ is sound and complete with respect to the class of all Topological Functional Frames.*

Furthermore, we characterize the subclasses of continuous, open and interior TFFs modally and axiomatize the corresponding bi-modal logics. Let us recall that a map $f : X_1 \to X_2$ is called *continuous* if the f-pre-images of τ_2-open sets are open in τ_1; the map is *open* if the images of open sets are open and the map is *interior* if it is both open and continuous.

Let us adopt the following notation for the corresponding bi-modal logics: let $S4_c$ denote the logic of all TFFs with f continuous; let $S4_o$ denote the logic of all TFFs with f open and let $S4_i$ denote the logic of all TFFs with f interior.

Proposition 8. *The three logics are axiomatized as follows:*

$$S4_c = S4_R + \big(\Diamonddot\Diamond p \to \Diamond\Diamonddot p\big)$$

$$S4_o = S4_R + \big(\Diamond\Diamonddot p \to \Diamonddot\Diamond p\big)$$

$$S4_i = S4_R + \big(\Diamonddot\Diamond p \leftrightarrow \Diamond\Diamonddot p\big)$$

and define their respective classes of TFFs.

Moreover, we show that

Proposition 9. *The logics $S4_R$ and $S4_c$ have the finite model property.*

From the fmp and the finite axiomatization we conclude that $S4_R$ and $S4_c$ logics are decidable. In fact, their decision problems are in the complexity class PSPACE.

The following literature was used:[1], [2], [3].

References

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