Category of Algebras view from Relational Structure Theory

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For an algebra A and an idempotent term operation e of A, the class mapping from the variety $\mathcal{V}(A)$ generated by A to $\mathcal{V}(A(e))$ defined as $C \mapsto C(e)$ is expanded to a functor between categories, where $C(e) = (e(C), \mathrm{Clo}(C(e)))$, $\mathrm{Clo}(C(e)) = \{(e \circ f)\restriction_{e(C)^m} \mid m \in \mathbb{N}, f \in \mathrm{Clo}_m(C)\}$. This functor is notable as a functor described by an interpretation from the language of $\mathcal{V}(A)$ to the language of $\mathcal{V}(A(e))$. Another important functor described by an interpretation is matrix power $\mathcal{V}(A) \to \mathcal{V}(A^{[n]})$, this functor is a categorical equivalence.

In relational structure theory, we consider the tuple of idempotent term operations $\overline{e} = (e_1, \dots, e_n)$ of an algebra A. For any tuple $\overline{e}$, we can consider a functor $\mathcal{V}(A) \to \mathcal{V}(A^{[n]}(\mathbf{e}))$ where $\mathbf{e}$ is an idempotent operation of $A^{[n]}$ which $(a_1, \dots, a_n)$ maps to $(e_1(a_1), \dots, e_n(a_n))$.

Definition 1. $\overline{e}$ is said to cover A if $C \mapsto C^{[n]}(\mathbf{e})$ is a categorical equivalence.

It is known that any finite algebras have unique minimal covers "up to isomorphism" [2]. Further, a minimal cover $(e_1, \dots, e_n)$ of an algebra A, each algebras $A(e_i)$ is irreducible (that is, it has no covers which do not contain the identity operation).

We call the structure of the matrix product $A^{[n]}(\mathbf{e})$, where $\overline{e}$ is a minimal cover of A, essential part of A. We denote the essential part of A $\mathrm{Ess}(A)$. Using this notion, categorical equivalence of locally finite varieties is characterized by the structure of the essential parts as following: For finite algebras A and B, there exists a categorical equivalence $\mathcal{V}(A) \to \mathcal{V}(B)$ which maps A to B if and only if the essential part of A is isomorphic to the essential part of B.

Furthermore, we prove that $\mathrm{Ess}(\mathrm{Ess}(A))$ is isomorphic to $\mathrm{Ess}(A)$ for any finite algebra A.

By these facts, we can consider the following framework classifying finite algebras.

1. Classifying irreducible finite algebras.
2. For any finite family of finite irreducible algebras, classifying the matrix products of the family.
3. Classifying categorically equivalent algebras for given algebra A which is isomorphic to $\mathrm{Ess}(A)$.

We will talk about this framework and related topics.

References

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