Brouwer's Fixed Point Theorem

An 'almost constructive' proof

Ezra Schoen

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Introduction

Brouwer

The Fixed Point Theorem

Theorem

Any continuous function from the closed (n-dimensional) disc to the closed (n-dimensional) disc has a fixed point; that is, a point that is left invariant by the function.

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If $f : \mathbb{D}^n \to \mathbb{D}^n$ is continuous, then there is an $x \in \mathbb{D}^n$ with $f(x) = x$.

A classical proof

A key lemma

Lemma

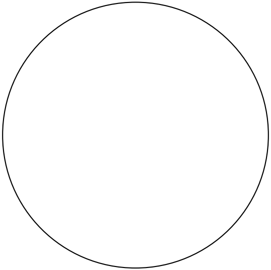
There is no continuous function $g : \mathbb{D}^2 \to \mathbb{S}^1$ such that $f(x) = x$ for all $x \in \mathbb{S}^1$.

Proof: If there were such an f, then $\mathbb{D}^2$ and $\mathbb{S}^1$ would be homotopy-equivalent, but $\mathbb{D}^2$ is contractible, whereas $\pi_1(\mathbb{S}^1) = \mathbb{Z} \neq 0$.

□

Constructing a function g

Assume $f : \mathbb{D}^2 \to \mathbb{D}^2$ has no fixed points.



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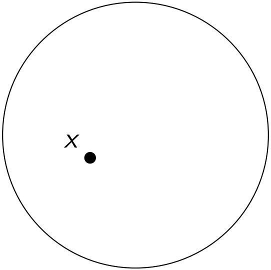

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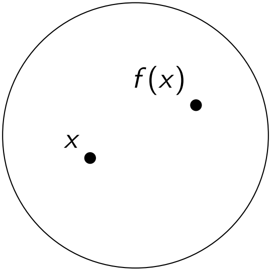

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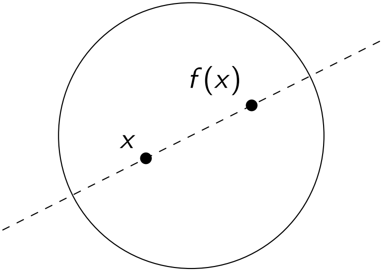

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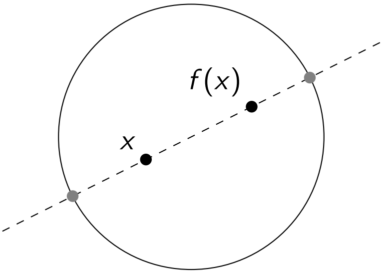

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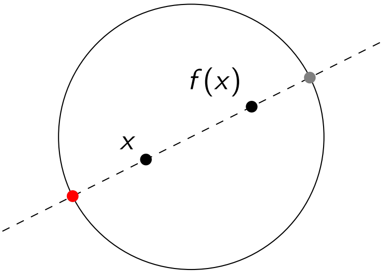

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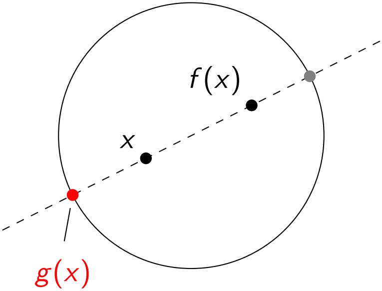

Finishing the proof

If $f : \mathbb{D}^2 \to \mathbb{D}^2$ has no fixed points, then there is a continuous $g : \mathbb{D}^2 \to \mathbb{S}^1$ with $g(x) = x$ for all $x \in \mathbb{S}^1$.

Finishing the proof

If $f : \mathbb{D}^2 \to \mathbb{D}^2$ has no fixed points, then there is a continuous $g : \mathbb{D}^2 \to \mathbb{S}^1$ with $g(x) = x$ for all $x \in \mathbb{S}^1$.Contradiction!

□

Unconstructivity

We can at best prove $\forall f : \neg\forall x \neg(f(x) = x)$. But this is not the same as $\forall f : \exists x(f(x) = x)$!

Sperner's lemma

The lemma

Lemma

Every Sperner coloring of a subdivision of the triangle has a rainbow triangle.

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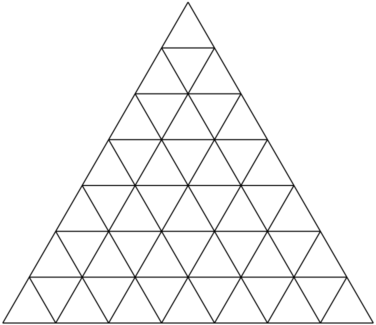

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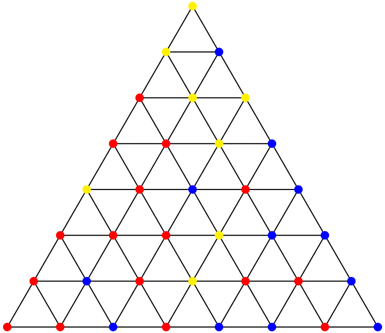

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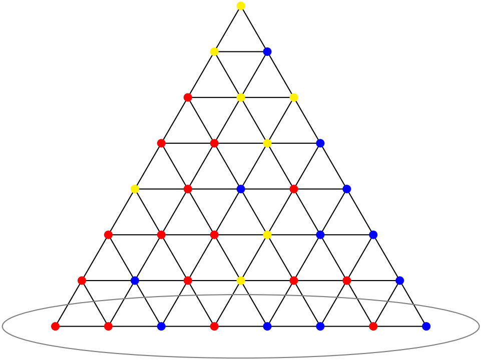

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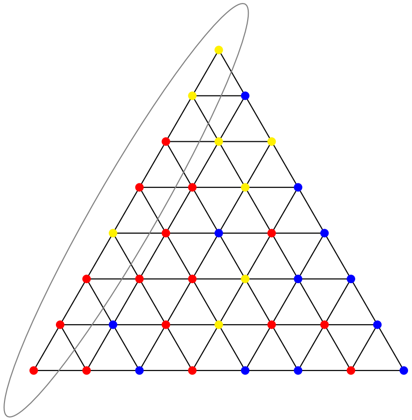

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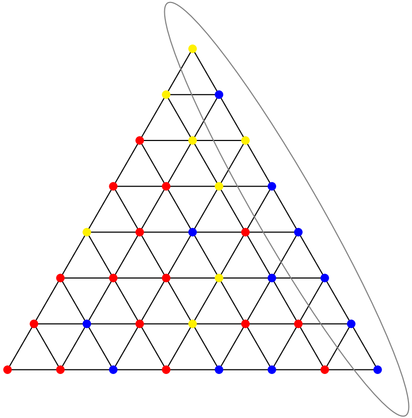

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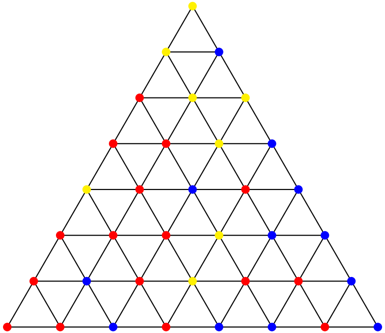

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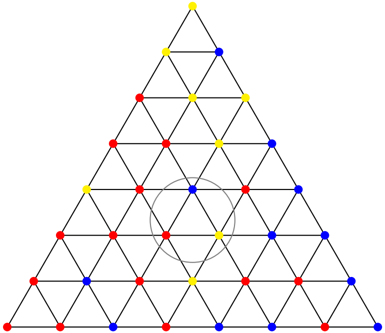

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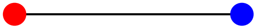

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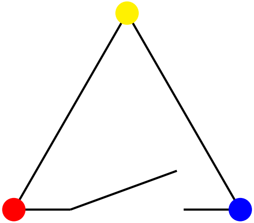

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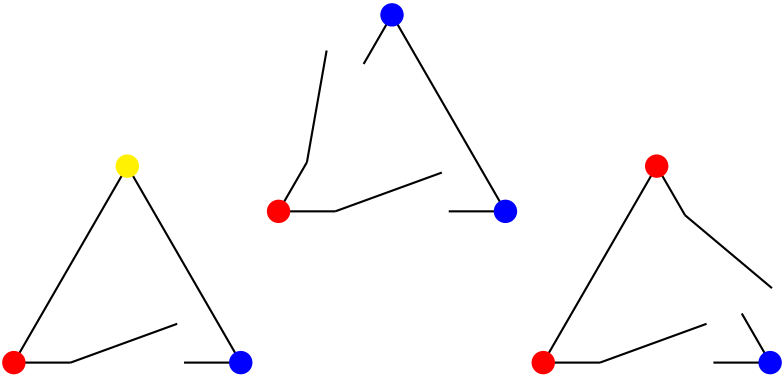
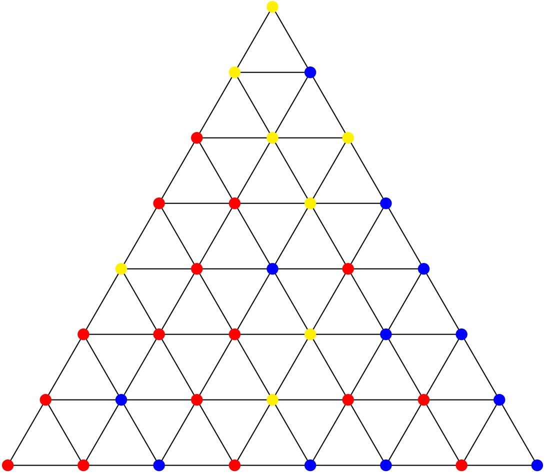


Proof of Sperner's Lemma (ii): Walks



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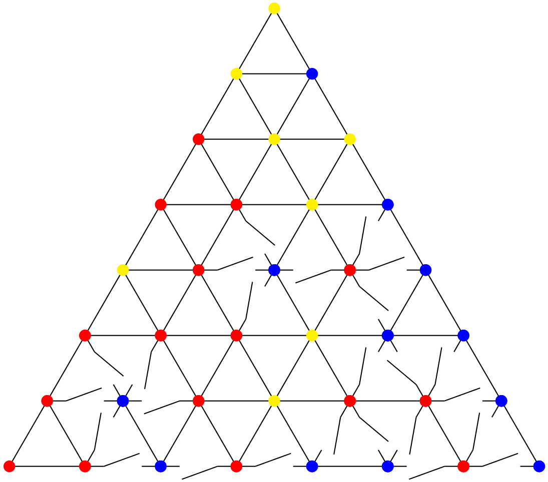

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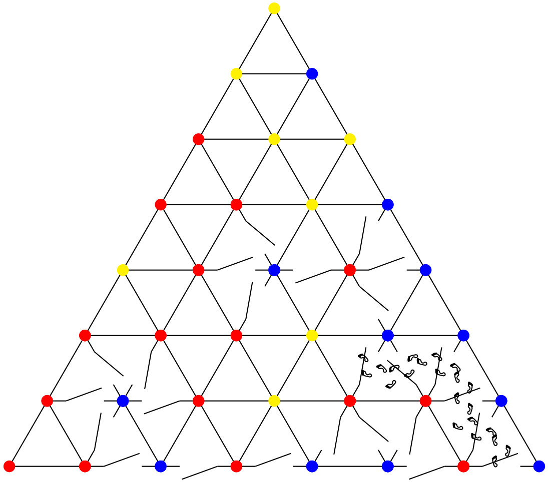

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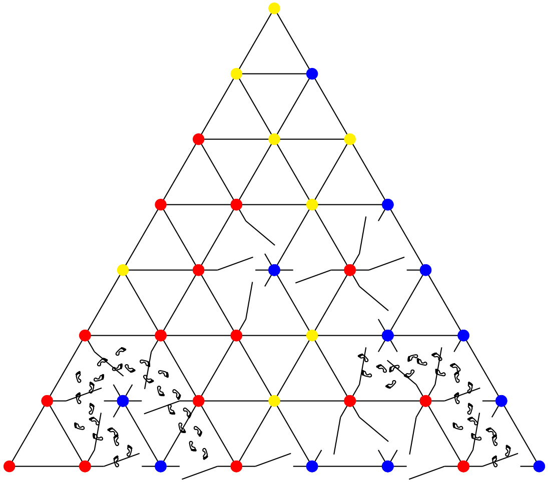

Proof of Sperner's lemma (iii): Counting Doors

Every exit either leads to a rainbow triangle, or is linked to one unique other exit.

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Proof: Go from left to right. The color changes after a segment if and only if it is an exit.

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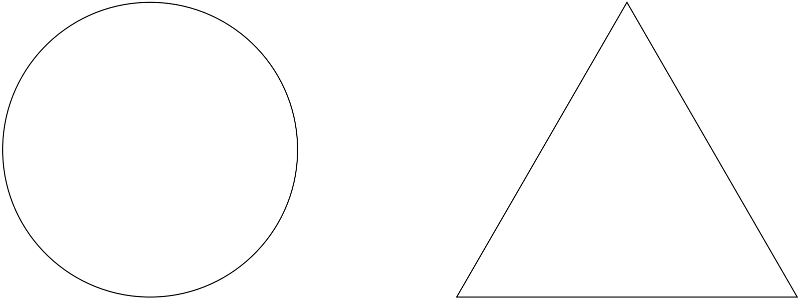
So at least one exit is not linked to another, hence there is a rainbow triangle.

Brouwer's fixed point theorem

A topological joke



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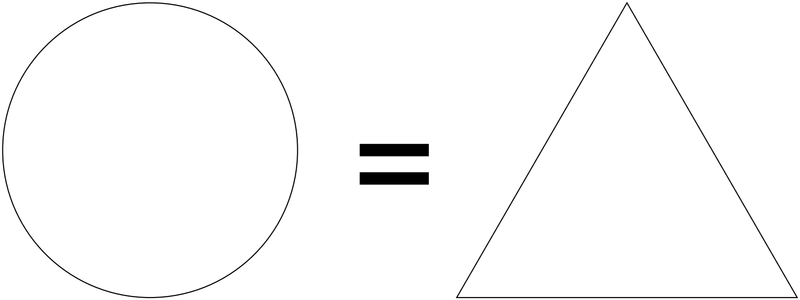

Coloring the standard simplex

$$\Delta^2 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 \mid t_1, t_2, t_3 \geq 0, t_1 + t_2 + t_3 = 1\}$$

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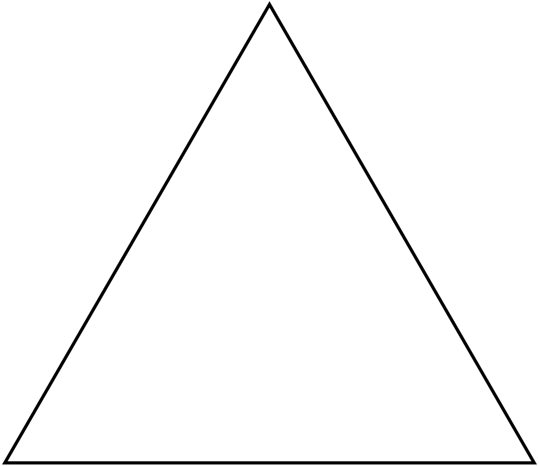
Goal

Find a shrinking sequence of rainbow triangles.

Further subdivisions



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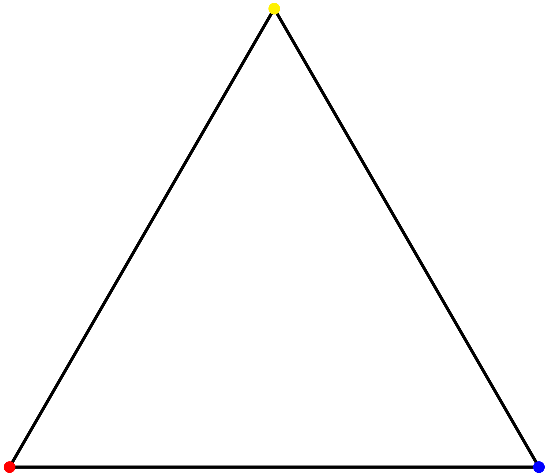

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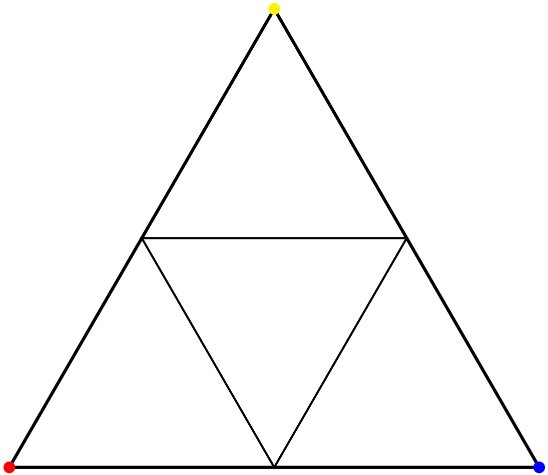

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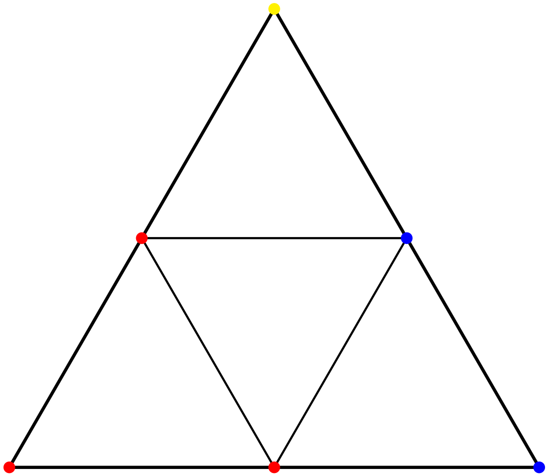

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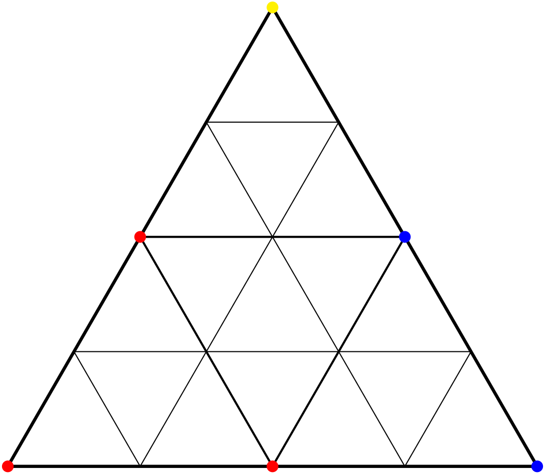

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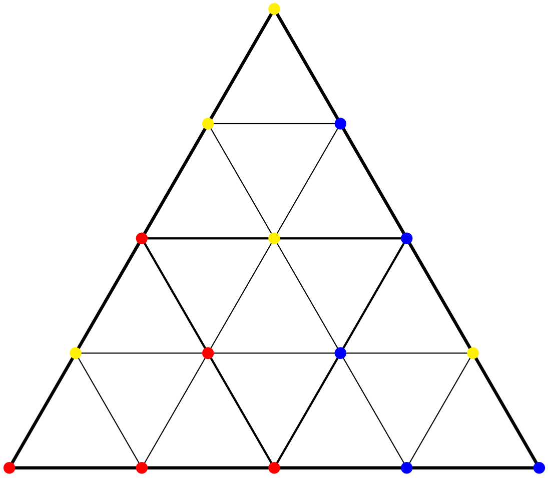

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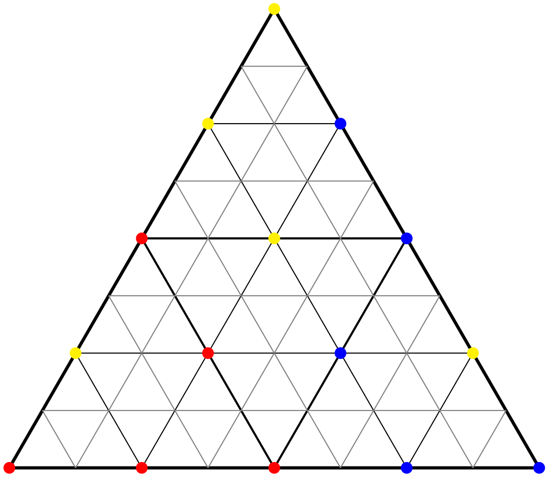

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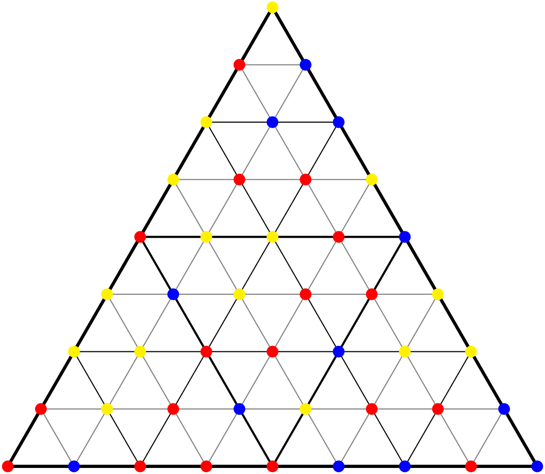
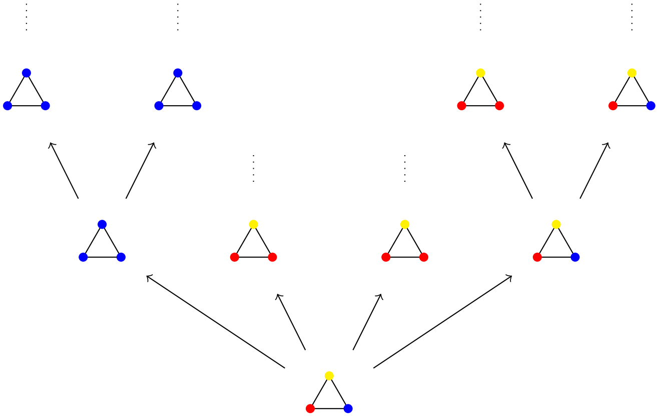


Building a tree



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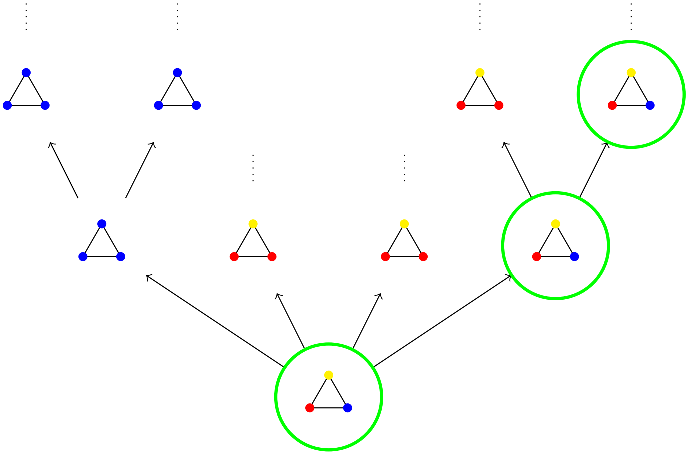

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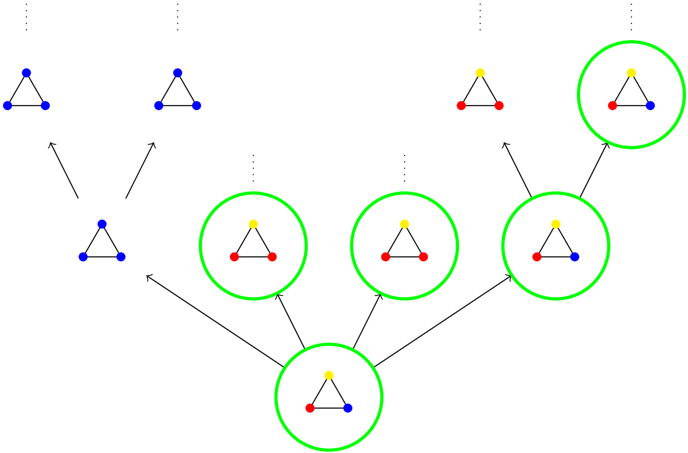

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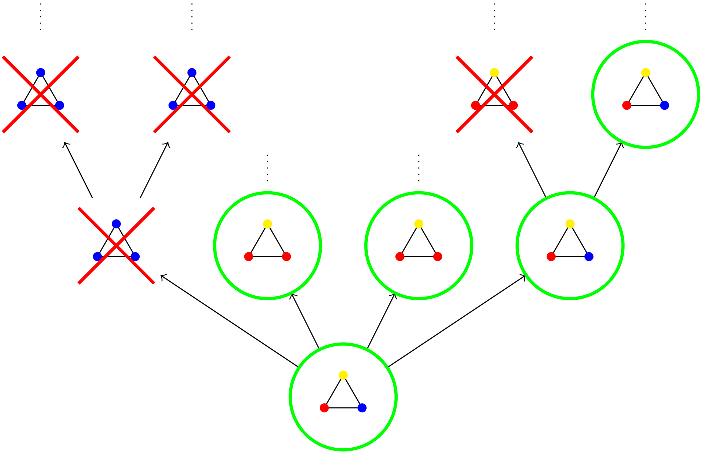
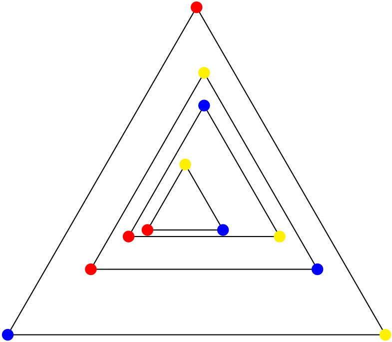


Finding the fixed point



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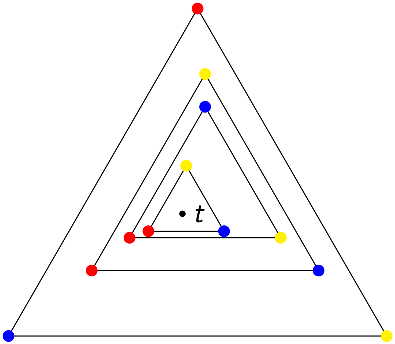

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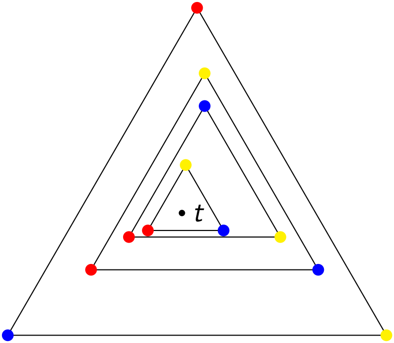

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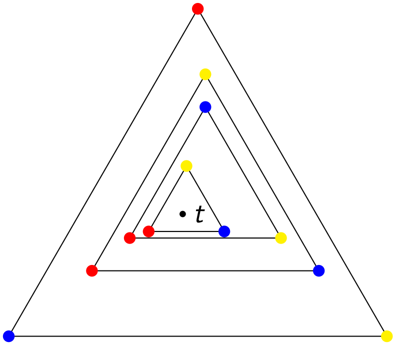

$\textcolor{red}{\bullet}_i \to t \quad \Rightarrow f(t)_1 \leq t_1$

$\textcolor{blue}{\bullet}_i \to t$

$\textcolor{yellow}{\bullet}_i \to t$

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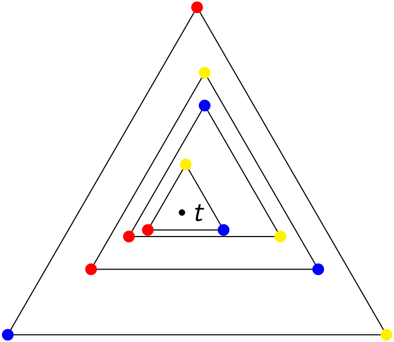

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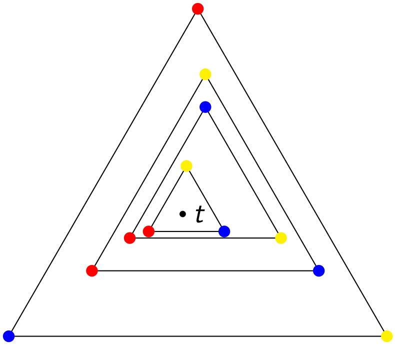

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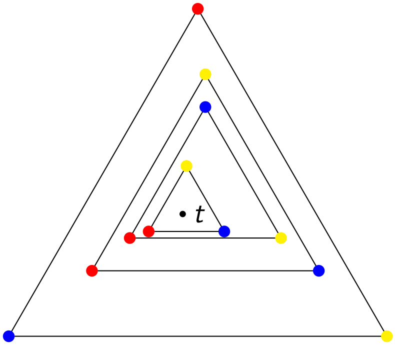

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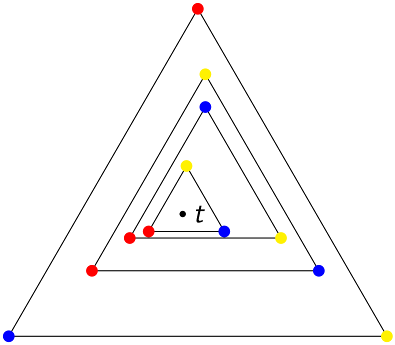
Finding the fixed point



$\bullet_i \to t \quad \Rightarrow f(t)_1 \leq t_1$ (red)

$\bullet_i \to t \quad \Rightarrow f(t)_2 \leq t_2$ (blue)

$\bullet_i \to t \quad \Rightarrow f(t)_3 \leq t_3$ (yellow)

$$t_1+t_2+t_3 = f(t)_1+f(t)_2+f(t)_3 (!)$$

So $t = f(t)$.

Constructive variants

Is the alternative proof constructive?

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NO

Is the alternative proof constructive?

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- Decidability of $\leq$

Is the alternative proof constructive?

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- Decidability of $\leq$ $\Leftarrow$ Let's not worry about this.

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- Choice principle!

Is the alternative proof constructive?

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Demand more

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Demand more or

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Demand more or **Promise less**

What choices do we make?

Weak Kőnig's lemma

If T is an infinite tree where every node has at most k children, then T has an infinite path.

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Fan theorem

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Fan theorem (Roughly)

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If T is an infinite tree where every node has at most k children, then T has an infinite path.

Fan theorem (Roughly)

If T is a tree where every node has at most k children, and there are no infinite paths in T, then T has finite depth $N < \omega$.

Demand more: Isolated fixed points

Theorem (Tanaka, 2011)

If $f : \Delta^2 \to \Delta^2$ is *uniformly* continuous, and *every fixed point is isolated*, then f has a fixed point.

Idea: If all fixed points are isolated, then in every infinite path, there is a stage after which we never have to make a choice. So by the Fan theorem, there is some *uniform bound* N on the number of necessary choices.

Promise less: Approximate fixed points

Theorem (Van Dalen, 2009)

If $f : \Delta^2 \to \Delta^2$ is *uniformly* continuous, then for every $\epsilon > 0$ there is an x with $|x - f(x)| < \epsilon$.

Idea: In a rainbow triangle, all the corners move in different directions. But they cannot be moved very far apart; so they must stay close to their original positions.

Thank you for listening!