The Computational Content of Classical Proofs
Extracting programs from classical proofs.

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Cool Logic
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Outline

1. Friedman’s $A$-translation

2. Program Extraction
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1. Friedman’s $A$-translation

2. Program Extraction
Kreisel’s theorem

Theorem (Kreisel (1958))

PA is a conservative extension of HA for $\Pi_2^0$-sentences.
Kreisel’s theorem

Theorem (Kreisel (1958))

PA is a conservative extension of HA for $\Pi^0_2$-sentences.

This means that

$$\vdash_{\text{PA}} \forall x \exists y . P(x, y) \iff \vdash_{\text{HA}} \forall x \exists y . P(x, y),$$

where $P$ is a computable predicate.

Corollary

A recursive function is provably total in Peano Arithmetic iff it is provably total in Heyting Arithmetic.
We first fix the language.

- $\mathcal{L}$ has logical constants $\bot, \land, \lor, \rightarrow, \forall, \exists$, variables $x, y, z, \ldots$, and binary predicate $=$.
- $\neg \varphi$ is an abbreviation of $\varphi \rightarrow \bot$. 
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- $\mathcal{L}$ has logical constants $\bot, \land, \lor, \rightarrow, \forall, \exists$, variables $x, y, z, \ldots$, and binary predicate $=$.

- $\neg \varphi$ is an abbreviation of $\varphi \rightarrow \bot$.

- Terms and formulas are defined as usual.

- $\vdash_C$ resp. $\vdash_I$ denotes classical resp. intuitionistic derivability in a natural deduction system.
Double-negation translation

Definition (Gödel, Gentzen)
Let $\varphi$ be a formula. Define the double-negation translation $\varphi^-$ of $\varphi$ as follows:

$$
\bot^- := \bot
$$
$$
\alpha^- := \neg\neg\alpha, \text{ where } \alpha \neq \bot \text{ is atomic}
$$
$$
(\varphi \lor \psi)^- := \neg\neg(\varphi^- \lor \psi^-)
$$
$$
(\varphi \land \psi)^- := \varphi^- \land \psi^{-}
$$
$$
(\varphi \rightarrow \psi)^- := \varphi^- \rightarrow \psi^{-}
$$
$$
(\forall x. \varphi)^- := \forall x. \varphi^{-}
$$
$$
\exists x. \varphi^- := \neg\neg\exists x. \varphi^{-}
$$

So $\varphi^-$ is the result of double-negating all atomic, disjunctive and existential subformulas of $\varphi$. 
Double-negation translation

Definition (Gödel, Gentzen)

Let $\varphi$ be a formula. Define the *double-negation translation* $\varphi^-$ of $\varphi$ as follows:

$\bot^-$ := $\bot$

$\alpha^-$ := $\neg\neg\alpha$, where $\alpha \neq \bot$ is atomic

$(\varphi \lor \psi)^-$ := $\neg(\varphi^- \lor \psi^-)$

$(\varphi \land \psi)^-$ := $\varphi^- \land \psi^-$

$(\varphi \rightarrow \psi)^-$ := $\varphi^- \rightarrow \psi^-$

$(\forall x.\varphi)^-$ := $\forall x.\varphi^-$

$\exists x.\varphi^-$ := $\neg\neg\exists x.\varphi^-$

So $\varphi^-$ is the result of double-negating all atomic, disjunctive and existential subformulas of $\varphi$. 
Some properties of the double-negation translation

Lemma

Let $\varphi$ be a formula, $\Gamma$ a set of formulas, and $\Gamma^- = \{ \psi^- \mid \psi \in \Gamma \}$.

1. $\vdash_C \varphi \leftrightarrow \varphi^-$,
2. $\neg \neg \varphi^- \vdash_I \varphi^-$,
3. If $\Gamma \vdash_C \varphi$, then $\Gamma^- \vdash_I \varphi^-$ (this justifies calling it a translation),
4. In general not $\varphi \vdash_I \varphi^-$. 
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Lemma

Let $\varphi$ be a formula, $\Gamma$ a set of formulas, and $\Gamma^- = \{\psi^- | \psi \in \Gamma\}$.

1. $\vdash C \varphi \leftrightarrow \varphi^-$,
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3. If $\Gamma \vdash C \varphi$, then $\Gamma^- \vdash I \varphi^-$ (this justifies calling it a translation),
4. In general not $\varphi \vdash_I \varphi^-.$

1, 2, and 3 are not very surprising, and their proofs are easy inductions on the depth of the derivation. 4 is less obvious. A counterexample is $\varphi = \neg\forall x. P(x)$. 
Friedman’s A-translation

Definition (Friedman)

Let $\varphi$ and $A$ be formulas such that no bound variable of $\varphi$ is free in $A$. We define the A-translation $\varphi^A$ of $\varphi$ as follows:

\[
\begin{align*}
\bot^A &:= A \\
\alpha^A &= \alpha \lor A, \text{ where } \alpha \neq \bot \text{ is atomic} \\
(\varphi \land \psi)^A &= \varphi^A \land \psi^A \\
(\varphi \lor \psi)^A &= \varphi^A \lor \psi^A \\
(\varphi \rightarrow \psi)^A &= \varphi^A \rightarrow \psi^A \\
(\forall x \varphi)^A &= \forall x \varphi^A \\
(\exists x \varphi)^A &= \exists x \varphi^A
\end{align*}
\]

So $\varphi^A$ is the result of substituting all atomic subformulas $\alpha$ with $\alpha \lor A$, and replacing any $\bot$ with $A$.

Note that $(\neg \alpha)^A = \alpha \lor A \rightarrow A$. 

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- $(\varphi \land \psi)^A := \varphi^A \land \psi^A$
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- $(\varphi \rightarrow \psi)^A := \varphi^A \rightarrow \psi^A$
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(\varphi \land \psi)^A & := \varphi^A \land \psi^A \\
(\varphi \lor \psi)^A & := \varphi^A \lor \psi^A \\
(\varphi \to \psi)^A & := \varphi^A \to \psi^A \\
(\forall x \varphi)^A & := \forall x \varphi^A \\
(\exists x \varphi)^A & := \exists x \varphi^A
\end{align*}
\]

So $\varphi^A$ is the result of substituting all atomic subformulas $\alpha$ with $\alpha \lor A$, and replacing any $\bot$ with $A$. Note that $(\neg \alpha)^A = \alpha \lor A \to A$. 

Some properties of Friedman’s $A$-translation

Lemma

Let $\varphi$ be formula, $\Gamma$ a set of formulas and $A$ a formula such that $\varphi^A$ and $\Gamma^A$ are defined, where $\Gamma^A = \{\psi^A | \psi \in \Gamma\}$.

1. $\vdash_C \varphi^A \leftrightarrow \varphi \lor A$
2. $A \vdash_I \varphi^A$
3. If $\Gamma \vdash_I \varphi$, then $\Gamma^A \vdash_I \varphi^A$
4. In general not $\varphi \vdash_I \varphi^A$

Proof of 1 and 2 are straight-forward inductions on the derivation. A counterexample of 4 is $\varphi := \neg \neg A$. 

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Proof of 1 and 2 are straight-forward inductions on the derivation. A counterexample of 4 is $\varphi := \neg \neg A$. 
Sketch of proof of 3: If $\Gamma \vdash I \varphi$, then $\Gamma^A \vdash I \varphi^A$

The rules $\land_I, \land_E, \lor_I, \lor_E, \rightarrow_I, \rightarrow_E$ are straightforward. See for example $\rightarrow_I$: 

\[
\begin{align*}
\Gamma, \psi \vdash \varphi &\Rightarrow \Gamma \vdash \varphi \\
\Gamma, \exists x.\varphi \vdash \varphi^{A} &\Rightarrow \Gamma^A \vdash \exists x.\varphi^{A}
\end{align*}
\]

because $\varphi^{A} = \varphi^{A}$.
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$$
\begin{array}{c}
\frac{D}{\Gamma, \varphi \vdash \psi} \\
\frac{\Gamma \vdash \varphi \rightarrow \psi}{\Gamma \vdash \varphi \rightarrow \psi} \rightarrow_I
\end{array}
$$
Sketch of proof of 3: If $\Gamma \vdash_I \varphi$, then $\Gamma^A \vdash_I \varphi^A$

The rules $\wedge_I, \wedge_E, \vee_I, \vee_E, \rightarrow_I, \rightarrow_E$ are straightforward. See for example $\rightarrow_I$:

$$
\frac{D}{\Gamma, \varphi \vdash \psi} \quad \rightarrow_I 
\quad \Rightarrow 
\frac{\ldots \quad \text{IH} \quad \ldots \ldots}{\Gamma^A, \varphi^A \vdash \psi^A} 
\quad \Rightarrow 
\frac{\Gamma^A \vdash \varphi^A \rightarrow \psi^A}{\Gamma^A \vdash \varphi^A \rightarrow \psi^A} \quad \rightarrow_I
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The rules $\land_I, \land_E, \lor_I, \lor_E, \rightarrow_I, \rightarrow_E$ are straightforward. See for example $\rightarrow_I$:

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\begin{array}{c}
\text{D} \\
\Gamma, \varphi \vdash \psi \\
\hline
\Gamma \vdash \varphi \rightarrow \psi \\
\end{array}
\rightarrow_I \\
\Rightarrow \\
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\text{IH} \\
\Gamma^A, \varphi^A \vdash \psi^A \\
\hline
\Gamma^A \vdash \varphi^A \rightarrow \psi^A \\
\end{array}
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$$

$\forall_I, \forall_E, \exists_I, \exists_E$ are a bit trickier because of variable bindings. We consider $\exists_I$:
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The rules $\wedge_I, \wedge_E, \vee_I, \vee_E, \to_I, \to_E$ are straightforward. See for example $\to_I$:

$$
\frac{D}{\Gamma, \varphi \vdash \psi} \quad \frac{\ldots \quad IH \quad \ldots \quad}{\Gamma^A, \varphi^A \vdash \psi^A} \\
\frac{\Gamma \vdash \varphi \to \psi}{\Gamma \vdash \varphi^A \to \psi^A}
$$

$\forall_I, \forall_E, \exists_I, \exists_E$ are a bit trickier because of variable bindings. We consider $\exists_I$:

$$
\frac{D}{\Gamma \vdash \varphi[t/x]} \quad \frac{\exists_I}{\Gamma \vdash \exists x. \varphi}
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The rules $\land_I, \land_E, \lor_I, \lor_E, \to_I, \to_E$ are straightforward. See for example $\to_I$:

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\frac{\varphi \to \psi}{\varphi \to \psi} \quad \implies \quad \frac{\varphi^A \to \psi^A}{\varphi^A \to \psi^A}
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$\forall_I, \forall_E, \exists_I, \exists_E$ are a bit trickier because of variable bindings. We consider $\exists_I$:

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\frac{\varphi[t/x]}{\exists x. \varphi} \quad \implies \quad \frac{\varphi^A[t/x]}{\exists x. \varphi^A}
\]

because $(\varphi[t/x])^A = \varphi^A[t/x]$ and $(\exists x. \varphi)^A = \exists x. \varphi^A$. 
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The rules $\land_I$, $\land_E$, $\lor_I$, $\lor_E$, $\to_I$, $\to_E$ are straightforward. See for example $\to_I$:

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$\forall_I$, $\forall_E$, $\exists_I$, $\exists_E$ are a bit trickier because of variable bindings. We consider $\exists_I$:

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\begin{array}{c}
\frac{D}{\Gamma \vdash \varphi[t/x]} \\
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\end{array}
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because $(\varphi[t/x])^A = \varphi^A[t/x]$ and $(\exists x. \varphi)^A = \exists x. \varphi^A$.

For $\bot_E$: IH is $\Gamma^A \vdash A$, and 2 gives us $A \vdash \varphi^A$. 
We add new symbols to the language:

- nullary constant 0,
- unary function symbol S,
- symbols $F, G, H, \ldots$ for all primitive recursive functions.
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- nullary constant $0$,
- unary function symbol $S$,
- symbols $F, G, H, \ldots$ for all primitive recursive functions.

Peano axioms:

1. **(refl)** $x = x$
2. **(trans)** $x = y \land y = z \rightarrow x = z$
3. **(cong$_F$)** $x_i = x'_i \rightarrow F(x_1, \ldots, x_i, \ldots, x_n) = F(x_1, \ldots, x'_i, \ldots, x_n)$ for any $n$-ary function constant $F$
4. **(succ$_1$)** $S(x) \neq 0$
5. **(succ$_2$)** $S(x) = S(y) \rightarrow x = y$
6. **(ind)** $\varphi(0) \land \forall x (\varphi(x) \rightarrow \varphi(S(x))) \rightarrow \forall x \varphi(x)$
We add new symbols to the language:
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Peano axioms:

- (refl) $x = x$
- (trans) $x = y \land y = z \rightarrow x = z$
- (cong) $x_i = x'_i \rightarrow F(x_1, \ldots, x_i, \ldots, x_n) = F(x_1, \ldots, x'_i, \ldots, x_n)$ for any $n$-ary function constant $F$
- (succ) $S(x) \neq 0$
- (succ) $S(x) = S(y) \rightarrow x = y$
- (ind) $\varphi(0) \land \forall x (\varphi(x) \rightarrow \varphi(S(x))) \rightarrow \forall x \varphi(x)$
- (proj) $F(x_1, \ldots, x_i, \ldots, x_n) = x_i$
- (comp) $F(x_1, \ldots, x_n) = G(H_1(x_1, \ldots, x_n), \ldots, H_m(x_1, \ldots, x_n))$
- (rec) $F(0, x_1, \ldots, x_n) = G(x_1, \ldots, x_n)$
  $\land F(S(y), x_1, \ldots, x_n) = H(F(y, x_1, \ldots, x_n), y, x_1, \ldots, x_n)$
Definition (Peano Arithmetic, Heyting Arithmetic)

Let $\Gamma$ be a subset of the Peano axioms and $\varphi$ be a formula.

- $\Gamma \vdash_{C} \varphi \iff \vdash_{PA} \varphi$
- $\Gamma \vdash_{I} \varphi \iff \vdash_{HA} \varphi$
Arithmetic

Definition (Peano Arithmetic, Heyting Arithmetic)

Let $\Gamma$ be a subset of the Peano axioms and $\varphi$ be a formula.

- $\Gamma \vdash C \varphi \implies \vdash_{PA} \varphi$
- $\Gamma \vdash I \varphi \implies \vdash_{HA} \varphi$

Fact

For any quantifier-free formula $\varphi(x_1, \ldots, x_n)$ there is a primitive recursive function symbol $F$ such that

$$\vdash_{HA} \varphi(x_1, \ldots, x_n) \iff F(x_1, \ldots, x_n) = 0.$$
Lemma

Let \( \varphi \) be a Peano axiom. Then \( \vdash_{HA} \varphi^- \) and \( \vdash_{HA} \varphi^A \).
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Let \( \varphi \) be a Peano axiom. Then \( \vdash_{HA} \varphi^- \) and \( \vdash_{HA} \varphi^A \).

Proof.
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Let $\varphi$ be a Peano axiom. Then $\vdash_{HA} \varphi^-$ and $\vdash_{HA} \varphi^A$.

Proof.

If $\varphi$ is on one of the forms

- $\alpha$,
- $\alpha \land \beta$,
- $\alpha \rightarrow \beta$ or
- $\alpha \land \beta \rightarrow \gamma$,

where $\alpha, \beta, \gamma$ are atomic, then $\varphi \vdash_{I} \varphi^-$ and $\varphi \vdash_{I} \varphi^A$. 
Lemma

Let $\varphi$ be a Peano axiom. Then $\vdash_{HA} \varphi^-$ and $\vdash_{HA} \varphi^A$.

Proof.

If $\varphi$ is on one of the forms

- $\alpha$,
- $\alpha \land \beta$,
- $\alpha \rightarrow \beta$ or
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where $\alpha, \beta, \gamma$ are atomic, then $\varphi \vdash_I \varphi^-$ and $\varphi \vdash_I \varphi^A$.

Luckily, everything, except instances of the induction scheme, is of this form.
Lemma

Let \( \varphi \) be a Peano axiom. Then \( \vdash_{HA} \varphi^- \) and \( \vdash_{HA} \varphi^A \).

Proof.

Let \( \varphi \) be an instance of the induction axiom:

\[
\varphi = \psi(0) \land \forall x(\psi(x) \rightarrow \psi(S(x))) \rightarrow \forall x.\psi(x),
\]

for some formula \( \psi(x) \).
Lemma

Let $\varphi$ be a Peano axiom. Then $\vdash_{\text{HA}} \varphi^-$ and $\vdash_{\text{HA}} \varphi^A$.

Proof.

Let $\varphi$ be an instance of the induction axiom:

$$\varphi = \psi(0) \land \forall x(\psi(x) \to \psi(S(x))) \to \forall x.\psi(x),$$

for some formula $\psi(x)$. Now:

$$\varphi^- = \psi^-(0) \land \forall x(\psi^-(x) \to \psi^-(S(x))) \to \forall x.\psi^-(x),$$
$$\varphi^A = \psi^A(0) \land \forall x(\psi^A(x) \to \psi^A(S(x))) \to \forall x.\psi^A(x),$$

which are themselves axioms of HA.
Corollary

1. If $\vdash_{PA} \varphi$, then $\vdash_{HA} \varphi^-$,

2. If $\vdash_{HA} \varphi$ and $\varphi^A$ is defined, then $\vdash_{HA} \varphi^A$.  

Proof.

1. Let $\Gamma$ be the axioms used in the derivation $\vdash_{PA} \varphi$.
   $\Gamma \vdash C \varphi = \Rightarrow \Gamma^- \vdash I \varphi^- = \Rightarrow \vdash_{HA} \varphi^-$.

2. Let $\Gamma$ be the axioms used in the derivation $\vdash_{HA} \varphi$.
   $\Gamma \vdash I \varphi = \Rightarrow \Gamma^A \vdash I \varphi^A = \Rightarrow \vdash_{HA} \varphi^A$. 

Corollary

1. If $\vdash_{PA} \varphi$, then $\vdash_{HA} \varphi^-$.

2. If $\vdash_{HA} \varphi$ and $\varphi^A$ is defined, then $\vdash_{HA} \varphi^A$.

Proof.

1. Let $\Gamma$ be the axioms used in the derivation $\vdash_{PA} \varphi$.

   $$\Gamma \vdash C \varphi \implies \Gamma^- \vdash I \varphi^- \implies \vdash_{HA} \varphi^-.$$
Corollary

1. If $\vdash_{PA} \varphi$, then $\vdash_{HA} \varphi^-$.
2. If $\vdash_{HA} \varphi$ and $\varphi^A$ is defined, then $\vdash_{HA} \varphi^A$.

Proof.

1. Let $\Gamma$ be the axioms used in the derivation $\vdash_{PA} \varphi$.

   $\Gamma \vdash_C \varphi \implies \Gamma^- \vdash_I \varphi^- \implies \vdash_{HA} \varphi^-$.

2. Let $\Gamma$ be the axioms used in the derivation $\vdash_{HA} \varphi$.

   $\Gamma \vdash_I \varphi \implies \Gamma^A \vdash_I \varphi^A \implies \vdash_{HA} \varphi^A$. 
Observation

If $\varphi$ is a $\sum^0_1$-formula, then $\vdash_I \varphi^A \leftrightarrow \varphi \vee A$.  

Proof.

$\exists y. \, F(x, y) = 0 \equiv \exists y. \, (F(x, y) = 0 \lor A)$

$\vdash_I \exists x. \, (\varphi \lor \psi) \leftrightarrow \exists x. \, \varphi \lor \psi$ when $x$ not free in $\psi$.

Therefore $\vdash_I \exists y. \, (F(x, y) = 0) \equiv \exists y. \, (F(x, y) = 0 \lor A)$.  

Friedman’s proof of Kreisel’s theorem
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Observation

If \( \varphi \) is a \( \Sigma^0_1 \)-formula, then \( \vdash_I \varphi^A \leftrightarrow \varphi \vee A \).

Proof.

\( (\exists y.F(x, y) = 0)^A = \exists y.(F(x, y) = 0 \vee A) \)
Observation

If $\varphi$ is a $\Sigma^0_1$-formula, then $\vdash_I \varphi^A \leftrightarrow \varphi \lor A$.

Proof.

- $(\exists y. F(x, y) = 0)^A = \exists y. (F(x, y) = 0 \lor A)$
- $\vdash_I \exists x (\varphi \lor \psi) \leftrightarrow \exists x \varphi \lor \psi$ when $x$ not free in $\psi$
Observation

If \( \varphi \) is a \( \Sigma^0_{1} \)-formula, then \( \vdash I \varphi^A \leftrightarrow \varphi \lor A \).

Proof.

- \( (\exists y. F(x, y) = 0)^A = \exists y. (F(x, y) = 0 \lor A) \)
- \( \vdash I \exists x (\varphi \lor \psi) \leftrightarrow \exists x \varphi \lor \psi \) when \( x \) not free in \( \psi \)
- Therefore \( \vdash I (\exists y. F(x, y) = 0)^A \leftrightarrow \exists y (F(x, y) = 0) \lor A \)
Friedman’s proof of Kreisel’s theorem

Proof of Theorem (Friedman).

- To show: $\vdash_{PA} \varphi \iff \vdash_{HA} \varphi$ for any $\Pi^0_2$-sentence $\varphi$.
- It is sufficient to show: $\vdash_{PA} \varphi \iff \vdash_{HA} \varphi$ for any $\Sigma^0_1$-formula.
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- It is sufficient to show: $\vdash_{\text{PA}} \varphi \iff \vdash_{\text{HA}} \varphi$ for any $\Sigma^0_1$-formula.
- Let $A := \exists y. F(x, y) = 0$. 
Friedman’s proof of Kreisel’s theorem

Proof of Theorem (Friedman).

- To show: $\vdash_{PA} \varphi \iff \vdash_{HA} \varphi$ for any $\Pi^0_2$-sentence $\varphi$.
- It is sufficient to show: $\vdash_{PA} \varphi \iff \vdash_{HA} \varphi$ for any $\Sigma^0_1$-formula.
- Let $A := \exists y. F(x, y) = 0$.
- Assume $\vdash_{PA} A$. 

Double-negation translation: $\vdash_{HA} \neg\neg A$.

Friedman’s $A$ translation: $\vdash_{HA} (\neg\neg A) A$.

$\vdash_{HA} (\neg\neg A) A \iff ((A \lor \neg A) \rightarrow A) \rightarrow A$.

$\vdash_{HA} A$. 

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Proof of Theorem (Friedman).

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- It is sufficient to show: \( \vdash_{\text{PA}} \varphi \iff \vdash_{\text{HA}} \varphi \) for any \( \Sigma^0_1 \)-formula.
- Let \( A := \exists y. F(x, y) = 0 \).
- Assume \( \vdash_{\text{PA}} A \).
- Double-negation translation: \( \vdash_{\text{HA}} \neg\neg A \).
- Friedman’s A translation: \( \vdash_{\text{HA}} (\neg\neg A)^A \).
Friedman’s proof of Kreisel’s theorem

Proof of Theorem (Friedman).

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- Let $A := \exists y. F(x, y) = 0$.
- Assume $\vdash_{\text{PA}} A$.
- Double-negation translation: $\vdash_{\text{HA}} \neg\neg A$.
- Friedman’s A translation: $\vdash_{\text{HA}} (\neg\neg A)^A$.
- $\vdash_{\text{HA}} (\neg\neg A)^A \iff (((A \lor A) \rightarrow A) \rightarrow A)$
Proof of Theorem (Friedman).

- To show: $\vdash_{PA} \varphi \iff \vdash_{HA} \varphi$ for any $\Pi^0_2$-sentence $\varphi$.
- It is sufficient to show: $\vdash_{PA} \varphi \iff \vdash_{HA} \varphi$ for any $\Sigma^0_1$-formula.
- Let $A := \exists y. F(x, y) = 0$.
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- Double-negation translation: $\vdash_{HA} \neg\neg A$.
- Friedman's $A$ translation: $\vdash_{HA} (\neg\neg A)^A$.
- $\vdash_{HA} (\neg\neg A)^A \iff (((A \lor A) \to A) \to A) \leftrightarrow A$. 

Friedman's proof of Kreisel's theorem
Proof of Theorem (Friedman).

- To show: \( \vdash_{\text{PA}} \varphi \iff \vdash_{\text{HA}} \varphi \) for any \( \Pi^0_2 \)-sentence \( \varphi \).
- It is sufficient to show: \( \vdash_{\text{PA}} \varphi \iff \vdash_{\text{HA}} \varphi \) for any \( \Sigma^0_1 \)-formula.
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- Double-negation translation: \( \vdash_{\text{HA}} \lnot \lnot A \).
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- \( \vdash_{\text{HA}} (\lnot \lnot A)^A \iff (((A \lor A) \rightarrow A) \rightarrow A) \iff A \).
- \( \vdash_{\text{HA}} A \).
Friedman’s $A$-translation

Program Extraction
Rice’s Theorem: It is in general undecidable whether a program meets some specification.

Proofs can easily be checked.

From a constructive proof, we can extract a correct program.
Rice’s Theorem: It is in general undecidable whether a program meets some specification.

Proofs can easily be checked.

From a constructive proof, we can extract a correct program.

\[
\vdash t : \forall x^A \exists y^B . P(x, y)
\]

\[
\varepsilon(t) : A \rightarrow B \\
\vdash \text{corr} : \forall x^A . P(x, (\varepsilon(t))(x))
\]
Example

- We want a sorting function $\text{sort} : \text{list}(\mathbb{N}) \to \text{list}(\mathbb{N})$. 
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Program Extraction II

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A perfect computer program: It does exactly what we want, and it is provably bug-free.
Using translations:

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Extraction from Classical Proofs I

Using translations:

$$\vdash_{\text{PA}} t : \forall x \exists y P(x, y) \quad \xrightarrow{\text{A-translation}} \quad \vdash_{\text{HA}} t' : \forall x \exists y P(x, y)$$

Double-negation translation, A-translation
Using translations:

\[ \vdash_{PA} t : \forall x \exists y P(x, y) \quad \overset{\text{A-translation}}{\longrightarrow} \quad \vdash_{HA} t' : \forall x \exists y P(x, y) \]

\[ f : \mathbb{N} \rightarrow \mathbb{N} \]

\[ f \text{ term in Gödel's System } T \]

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Using translations:

\[ \Gamma \vdash PA \vdash t : \forall x \exists y P(x, y) \]

\[ \rightarrow \]

\[ \Gamma \vdash HA \vdash t' : \forall x \exists y P(x, y) \]

\[ g : N \rightarrow N \]

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Intuitionistic proofs:
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- Control operators allow for more flexibility; it compares to adding labels and jumps, `return` or exception handling.
Extraction from Classical Proofs II

- **Intuitionistic proofs:**
  - Extracts *pure functional* programs.

- **Classical proofs:**
  - Needs a more expressive programming language.
  - Griffin (1990): Classical reasoning corresponds to *control operators*.
  - Control operators allow for more flexibility; it compares to adding labels and jumps, *return* or exception handling.

- Underlying algorithms in classical proofs are potentially more efficient than ones from intuitionistic proofs.
A traditional functional program $\text{mult} : \text{list}(\text{N}) \rightarrow \text{N}$ would have a computation similar to this:

$$\text{mult}[5, 7, 0, 2] \mapsto$$
Programs with control operators

- A traditional functional program \( \text{mult} : \text{list}(\mathbb{N}) \rightarrow \mathbb{N} \) would have a computation similar to this:

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\text{mult}[5, 7, 0, 2] \mapsto 5 \cdot (\text{mult}[7, 0, 2])
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Double negation translation $\leftrightarrow$ CPS-translation

- CPS: Continuation Passing Style
- CPS style function: The control appears explicitly in the form of a \textit{continuation} that is passed to the function.
Double negation translation $\leftrightarrow$ CPS-translation
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Instead, we want to extract to a system that has control as a primitive construct.

One approach is to interpret classical logics in a control calculus via a Curry-Howard correspondence (proofs-as-terms).
- This requires a lot of fiddling around with reduction strategies. And program extraction tend to not necessarily be correct.
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Another approach is realisability.

- Realisability can be seen as a formalisation of the BHK-interpretation: A realiser of an existential formula gives a witness for the formula, and a realiser of a disjunction tells which side of the disjunction is provable.
Which fragment of classical logic should we consider?

- EM₁: Excluded middle restricted to $\Sigma^0_1$-formulas.
- Markov’s Principle: $\neg\neg\exists x P(x) \rightarrow \exists x P(x)$
EM₁: Alwayz into somethin’

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- A natural place to start seems to be HA + EM₁
  - HA + EM₁ proves a lot of theorems (Akama, Berardi, Hayashi, Kohlenbach 2004)
EM$_1$: Alwayz into somethin’

- Which fragment of classical logic should we consider?
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- A natural place to start seems to be HA + EM$_1$
  - HA + EM$_1$ proves a lot of theorems (Akama, Berardi, Hayashi, Kohlenbach 2004)

- Traditional realisability cannot be used for HA + EM$_1$:
  - HA + EM$_1$ ⊢ $\forall x \forall y (\exists z Txyz \lor \forall z \neg Txyz)$, where $T$ is Kleene’s predicate.
  - A (traditional) realiser of this would solve the Halting Problem.
Aschieri's Interactive Learning-Based Realisability is based on the idea of learning by counterexamples.

- Knowledge states $S$.
- At any state $s$, we have a truth value of all instances $\exists y P(x, y) \lor \forall y \neg P(x, y)$ of $EM_1$, and in case of $\exists y P(x, y)$ being “true”, also a witness $m$. 

A learning-based realiser is a self-correcting program.
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- The realiser learns:
  - At stage $s$: It believes $\forall x \neg P(x)$
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Since a proof is finite, we only need a finite piece of information about $EM_1$.

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I will investigate whether we from $HA + EM_1$-proofs of $\Pi_0^2$-sentences can extract programs that uses control.
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Thank you!
Counterexample to 4: In general not $\varphi \vdash_I \varphi^-$. 

Consider a Kripke model with $\omega$ many nodes $k_0 \leq k_1 \leq k_2 \leq \ldots$, with the following domains and valuations.

\[
\begin{array}{c|cccc}
  i & 0 & 1 & 2 & \ldots \\
  \hline
  D(k_i) & \{0\} & \{0, 1\} & \{0, 1, 2\} & \ldots \\
  P & \{\} & \{0\} & \{0, 1\} & \ldots \\
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Hence $k_0 \models \forall x. \neg\neg P(x)$. This proves that we cannot have $\neg\forall x. P(x) \vdash_I \neg\neg P(x)$. 

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