An inquisitive semantics with witnesses*

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1 Introduction

The main purpose of inquisitive semantics is to enrich the notion of semantic meaning in such a way that it does not only embody informative content, but also inquisitive content. That is, the meaning assigned to a sentence should not only determine the information that the sentence provides, but also the issue that it raises, and the range of responses that would resolve this issue. Such a notion of meaning, and a semantics that assigns such meanings to sentences in the language of first-order logic, has been developed and investigated in quite some detail in recent work [2,4,6,9]. The resulting system, Inq_A, deals with informative and inquisitive content in a satisfactory way.

A second important goal of inquisitive semantics is to establish a suitable logical notion of *compliance*, which judges whether an utterance in a conversation is logically related to what was said before. Such a notion has been proposed in [6]. As long as we restrict our attention to the language of propositional logic, this notion seems to give satisfactory results. However, problems arise in the first-order case [2,3]. These problems do not just show that there is something wrong with the particular notion of compliance proposed in [6]. Rather, they reveal that certain first-order sentences which intuitively have a different range of compliant responses are entirely equivalent in Inq_A, which means that the very notion of meaning adopted in Inq_A is not fine-grained enough to serve as a basis for a suitable notion of compliance in the first-order setting [2,3].

Thus, Inq_A fulfills our primary purpose, in that it suitably deals with informative and inquisitive content. However, it does not achieve the second important goal, in that it is too coarse-grained to deal with compliance. In this paper we develop a more fine-grained inquisitive semantics, Inq_W, which is intended to overcome this limitation.

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2 Background

We start with a brief recapitulation of Inq_A. We will first consider the language of propositional logic, and then move on to the first-order setting. More elaborate expositions of Inq_A can be found in [2,4,6,9].

2.1 Propositional InqA

In this section we consider a language $\mathcal{L}_\mathcal{P}$, whose expressions are built up from $\bot$ and a finite set of proposition letters $\mathcal{P}$, using the binary connectives $\wedge, \vee$ and $\rightarrow$. We use $\neg\varphi$ as an abbreviation of $\varphi \rightarrow \bot$, $!\varphi$ as an abbreviation of $\neg\neg\varphi$, and $?\varphi$ as an abbreviation of $\varphi \vee \neg\varphi$. We refer to $!\varphi$ and $?\varphi$ as the non-inquisitive and the non-informative projection of φ, respectively.

Definition 1 (Worlds).
A world is a function from $\mathcal{P}$ to $\{0, 1\}$. We denote by W the set of all worlds.

Definition 2 (States).
A state is a set of worlds. We denote by $\mathcal{S}$ the set of all states.

Definition 3 (Support).

$$
\begin{array}{lll}
s \models p & \textit{iff} & \forall w \in s : w(p) = 1 \\
s \models \bot & \textit{iff} & s = \emptyset \\
s \models \varphi \wedge \psi & \textit{iff} & s \models \varphi \textit{ and } s \models \psi \\
s \models \varphi \vee \psi & \textit{iff} & s \models \varphi \textit{ or } s \models \psi \\
s \models \varphi \rightarrow \psi & \textit{iff} & \forall t \subseteq s : \textit{if } t \models \varphi \textit{ then } t \models \psi
\end{array}
$$

It follows from the above definition that the empty state supports any sentence φ. Thus, we may think of $\emptyset$ as the *absurd* state.

Fact 1 (Persistence) *If $s \models \varphi$ then for every $t \subseteq s$: $t \models \varphi$*

Fact 2 (Singleton states behave classically) *For any w and φ:*

$$\{w\} \models \varphi \quad \Longleftrightarrow \quad w \models \varphi \ \textit{ in classical propositional logic}$$

It can be derived from definition 3 that the support-conditions for $\neg\varphi$, $!\varphi$, and $?\varphi$ are as follows.

Fact 3 (Support for negation and the projection operators)

1. $s \models \neg\varphi$ *iff* $\forall w \in s : w \not\models \varphi$
2. $s \models !\varphi$ *iff* $\forall w \in s : w \models \varphi$
3. $s \models ?\varphi$ *iff* $s \models \varphi$ *or* $s \models \neg\varphi$

In terms of support, we define the *proposition* expressed by a sentence φ, and the *truth-set* of φ. The latter is the meaning that would be associated with φ in classical propositional logic.

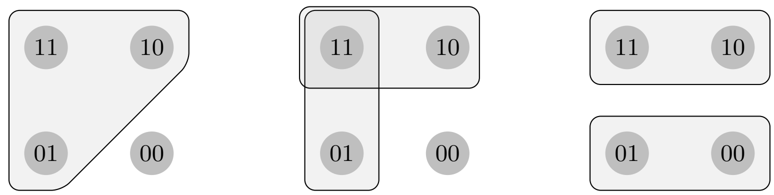

Fig. 1. (a) classical picture of $p \vee q$, (b) inquisitive picture of $p \vee q$, and (c) polar question $?p$.

Definition 4 (Propositions, entailment, and equivalence).

- $[\varphi] \quad := \quad \{s \in \mathcal{S} \mid s \models \varphi\}$
- $\varphi \models \psi$ *iff* *for all* s*: if* $s \models \varphi$*, then* $s \models \psi$
- $\varphi \equiv \psi$ *iff* $\varphi \models \psi$ *and* $\psi \models \varphi$

We will refer to the maximal elements of $[\varphi]$ as the *alternatives* for φ.

Definition 5 (Alternatives). *Let* φ *be a sentence.*

1. *Every maximal element of* $[\varphi]$ *is called an* alternative *for* φ.
2. *The* alternative set *of* φ, $[\![\varphi]\!]$, *is the set of alternatives for* φ.

The following result guarantees that the alternative set of a sentence completely determines the proposition that the sentence expresses, and vice versa.

Fact 4 (Propositions and alternatives) *For any state* s *and sentence* φ*:*

$$s \in [\varphi] \iff s \textit{ is contained in some } \alpha \in [\![\varphi]\!]$$

Example 1 (Disjunction). Inquisitive semantics differs from classical semantics in its treatment of disjunction. To see this, consider figures 1(a) and 1(b). In these figures, it is assumed that $\mathcal{P} = \{p, q\}$; world 11 makes both p and q true, world 10 makes p true and q false, etcetera. Figure 1(a) depicts the truth set—that is, the classical meaning—of $p \vee q$: the set of all worlds that make at least one of p and q true. Figure 1(b) depicts the alternative set of $p \vee q$ in Inq_A. It consists of two alternatives. One alternative is made up of all worlds that make p true, and the other of all worlds that make q true.

We think of a sentence φ as expressing a proposal to update the common ground of a conversation—formally conceived of as a set of possible worlds—in such a way that the new common ground supports φ. In other words, given fact 4, a sentence proposes to update the common ground in such a way that the resulting common ground is contained in one of the alternatives for φ.

Worlds that are not contained in any state supporting φ will not survive any of the updates proposed by φ. In other words, if any of the updates proposed by φ is executed, all worlds that are not contained in $\bigcup[\varphi]$ will be eliminated. Therefore, we refer to $\bigcup[\varphi]$ as the *informative content* of φ.

Definition 6 (Informative content). $\mathit{info}(\varphi) := \bigcup[\varphi]$

Classically, the informative content of φ is captured by the set of all worlds in which φ is classically true. We refer to this set of worlds as the *truth-set* of φ.

Definition 7 (Truth sets).
The truth set of φ, $|\varphi|$, is the set of all worlds where φ is classically true.

The following result says that, as far as informative content goes, Inq_A does not diverge from classical propositional logic. In this sense, Inq_A is a conservative extension of classical propositional logic.

Fact 5 (Informative content is classical) *For any φ:* $\mathit{info}(\varphi) = |\varphi|$

A sentence φ is informative in a state s iff it proposes to eliminate at least one world in s, i.e., iff $s \cap \mathsf{info}(\varphi) \neq s$. On the other hand, φ is inquisitive in s iff in order to reach a state $s' \subseteq s$ that supports φ it is not enough to incorporate the informative content of φ itself into s, i.e., $s \cap \mathsf{info}(\varphi) \not\models \varphi$, which means that φ requests a response from other participants that provides additional information.

Definition 8 (Inquisitiveness and informativeness in a state).

- φ *is* informative *in s iff* $s \cap \mathit{info}(\varphi) \neq s$
- φ *is* inquisitive *in s iff* $s \cap \mathit{info}(\varphi) \not\models \varphi$

Besides these notions of informativeness and inquisitiveness *relative to a state* we may also define absolute notions of informativeness and inquisitiveness.

Definition 9 (Absolute inquisitiveness and informativeness).

- φ *is* informative *iff it is informative in at least one state.*
- φ *is* inquisitive *iff it is inquisitive in at least one state.*

Fact 6 (Inquisitiveness, informativeness, and informative content)

- φ *is* informative *iff* $\mathit{info}(\varphi) \neq W$
- φ *is* inquisitive *iff* $\mathit{info}(\varphi) \not\models \varphi$

Fact 7 (Inquisitiveness and alternatives)

- φ *is* inquisitive *iff* $[\![\varphi]\!]$ *contains at least two alternatives.*

Example 2 (Disjunction continued). As in the classical setting, $p \vee q$ is *informative*, in that it proposes to eliminate worlds where both p and q are false. But it is also *inquisitive*, in that it proposes to move to a state that supports p or to a state that supports q, while merely eliminating worlds where both p and q are false is not sufficient to reach such a state. Thus, $p \vee q$ requests a response that provides additional information. This inquisitive aspect of meaning is not captured in the classical setting.

Definition 10 (Questions, assertions, and hybrids).

– φ *is a* question *iff it is not informative;*
– φ *is an* assertion *iff it is not inquisitive;*
– φ *is a* hybrid *iff it is both informative and inquisitive.*

Example 3 (Questions, assertions, and hybrids). We saw above that $p \vee q$ is both informative and inquisitive, i.e., hybrid. Figure 1(a) depicts the alternative set of $!(p \vee q)$, which consists of exactly one alternative. So $!(p \vee q)$ is an assertion. Figure 1(c) depicts the alternative set of $?p$. Together the alternatives for $?p$ cover the entire logical space, so $?p$ does not propose to eliminate any world. That is, $?p$ is a question.[3]

2.2 Compliance

Just as entailment traditionally judges whether an argument is valid, the logical notion of compliance is intended to judge whether a certain conversational move is related to the foregoing discourse. For our purposes in this paper, it is sufficient to consider a specific type of compliant responses to a given initiative. We will refer to these responses as *basic* compliant responses. For discussion of a more general notion of compliance we refer to [6].

Intuitively, a basic compliant response to an initiative φ is an assertion that resolves the issue that is raised by φ without providing more information than necessary. Formally, it is defined as follows:

Definition 11 (Basic compliant responses).
ψ is a basic compliant response to φ iff $[\![\psi]\!] = \{\alpha\}$ for some $\alpha \in [\![\varphi]\!]$.

Recall that the alternatives for φ are *maximal* supporting states. A response to φ is issue-resolving just in case it provides enough information to establish a state that supports φ. Thus, issue-resolving responses that do not provide more information than necessary supply exactly enough information to establish a *maximal* state that supports φ. This explains why basic compliant responses are defined in terms of alternatives, i.e., maximal supporting states.

To illustrate the notion of basic compliant responses, consider the question in (1) and the responses in (1-a-c).

(1) Is Mary going to the party?
  a. Yes, she is going.
  b. Cats don't like broccoli.

[3] Notice that questions, as defined here, are not necessarily inquisitive, and assertions are not necessarily informative. For instance, the tautology $!(p \vee \neg p)$ is both a question and an assertion, even though (or rather *because*) it is neither inquisitive nor informative. It is possible to give a slightly more involved definition of questions and assertions, which makes sure that the two notions are strictly disjoint (see [6]). This may be more desirable from a linguistic point of view, but the additional complexity is not quite relevant in the present setting, and is therefore avoided.

c. Yes, she is going, and cats don't like broccoli.

(1-a) is a basic compliant response to (1), but (1-b) and (1-c) are not. (1-b) is not compliant because it does not resolve the issue raised by (1), and (1-c) is not compliant because it provides *more* information than is needed to resolve the issue raised by (1). Only (1-a) provides exactly enough information to resolve the given issue.

As long as we restrict ourselves to the language of propositional logic, the notion of basic compliant responses (and the more general notion of compliance that it is a particular instance of) yield satisfactory results. However, we will see right below that this is no longer the case if we move to the first-order setting.

2.3 First-order InqA

Let $\mathcal{L}$ be a first-order language. The worlds that make up a *state* will now be first-order models for $\mathcal{L}$. We will assume that all worlds in a state share the same domain and the same interpretation of individual constants and function symbols. This assumption is enacted using the notion of a discourse model.

Definition 12 (Discourse models).
A discourse model $\mathbb{D}$ for $\mathcal{L}$ is a pair $\langle D, I\rangle$, where D is a domain and I an interpretation of all individual constants and function symbols in $\mathcal{L}$.

Definition 13 ($\mathbb{D}$-worlds and $\mathbb{D}$-states).
Let $\mathbb{D} = \langle D, I\rangle$ be a discourse model for $\mathcal{L}$. Then:

- *A $\mathbb{D}$-world w is a model $\langle D_w, I_w\rangle$ such that $D_w = D$ and I_w coincides with I as far as individual constants and function symbols are concerned. The set of all $\mathbb{D}$-worlds is denoted by $W_{\mathbb{D}}$.*
- *A $\mathbb{D}$-state is a set of $\mathbb{D}$-worlds. The set of all $\mathbb{D}$-states is denoted by $\mathcal{S}_{\mathbb{D}}$.*

Thus, a $\mathbb{D}$-state s is a set of first-order models for $\mathcal{L}$ that are all based on the same discourse model $\mathbb{D}$. This means that all the models in s share the same domain, and assign the same interpretation to individual constants and function symbols. The interpretation of *predicate symbols* is not fixed by $\mathbb{D}$, and may therefore differ from model to model in s.

The rationale behind the notion of discourse models and $\mathbb{D}$-states is that we make the (simplifying but not necessarilly realistic) assumption that the domain of discourse and the interpretation of the individual constants and function symbols is common knowledge among all the participants of a conversation. Thus, the information states of the individual participants, as well as the information state that represents the common ground of the conversation, are all based on the same discourse model, and the exchange of information only concerns the denotation of the predicate symbols.

The definitions below assume a fixed discourse-model $\mathbb{D} = \langle D, I\rangle$ for $\mathcal{L}$. Moreover, for any assignment g, we denote by $|\varphi|_g$ the *truth set* of φ relative to g, i.e., the set of worlds w such that $w \models_g \varphi$ in classical first-order logic.

Definition 14 (Support in first-order Inq_A).
Let s be a $\mathbb{D}$-state, g an assignment, and φ a formula in $\mathcal{L}$.

$$\begin{array}{lll}
s \models_g \varphi & \textit{iff} & s \subseteq |\varphi|_g \quad \textit{for atomic } \varphi \\
s \models_g \bot & \textit{iff} & s = \emptyset \\
s \models_g \varphi \wedge \psi & \textit{iff} & s \models_g \varphi \textit{ and } s \models_g \psi \\
s \models_g \varphi \vee \psi & \textit{iff} & s \models_g \varphi \textit{ or } s \models_g \psi \\
s \models_g \varphi \to \psi & \textit{iff} & \forall t \subseteq s : \textit{if } t \models_g \varphi \textit{ then } t \models_g \psi \\
s \models_g \forall x.\varphi & \textit{iff} & s \models_{g[x/d]} \varphi \textit{ for all } d \in D \\
s \models_g \exists x.\varphi & \textit{iff} & s \models_{g[x/d]} \varphi \textit{ for some } d \in D
\end{array}$$

Definition 15 (Propositions, entailment, and equivalence).

- $[\varphi]_g \quad := \quad \{s \in \mathcal{S}_{\mathbb{D}} \mid s \models_g \varphi\}$
- $\varphi \models \psi$ *iff for all s and g: if $s \models_g \varphi$, then $s \models_g \psi$*
- $\varphi \equiv \psi$ *iff $\varphi \models \psi$ and $\psi \models \varphi$*

All the basic logical notions defined in the propositional setting, like informativeness, inquisitiveness, questions, assertions, and hybrids, carry over immediately to the first order setting. Moreover, all the basic properties of the system having to do with informative and inquisitive content still hold. For instance, the classical treatment of informative content is still preserved (fact 2).

However, one feature of the system is not preserved: the proposition expressed by a sentence is no longer fully determined by the alternative set of that sentence (fact 4). In other words, it is no longer the case that every state supporting φ is contained in a maximal state supporting φ. In fact, as shown by Ciardelli [2,3], there are first-order formulas that do not have any maximal supporting states.

Example 4 (The boundedness formula). Consider a language which has a unary predicate symbol P, a binary function symbol $+$, and the set $\mathbb{N}$ of natural numbers as its individual constants. Consider the discourse-model $\mathbb{D} = \langle D, I\rangle$, where $D = \mathbb{N}$, I maps every $n \in \mathbb{N}$ to itself, and $+$ is interpreted as addition. Let $x \leq y$ abbreviate $\exists z(x + z = y)$, let $B(x)$ abbreviate $\forall y(P(y) \to y \leq x)$, and for every $n \in \mathbb{N}$, let $B(n)$ abbreviate $\forall y(P(y) \to y \leq n)$. Intuitively, $B(n)$ says that n is greater than or equal to any number in P. In other words, $B(n)$ says that n is an *upper bound* for P.

A $\mathbb{D}$-state s supports a formula $B(n)$, for some $n \in \mathbb{N}$, if and only if $B(n)$ is true in every world in s, that is, if and only if n is an upper bound for P in every w in s. Now consider the formula $\exists x.B(x)$, which intuitively says that there is an upper bound for P. This formula, which Ciardelli refers to as the *boundedness formula*, does not have a maximal supporting state. To see this, let s be an arbitrary state supporting $\exists x.B(x)$. Then there must be a number $n \in \mathbb{N}$ such that s supports $B(n)$, i.e., $B(n)$ must be true in all worlds in s. Now let w^* be the $\mathbb{D}$-world in which P denotes the singleton set $\{n + 1\}$. Then w^* cannot be in s, because it does not make $B(n)$ true. Thus, the state s^* which is obtained from s by adding w^* to it is a proper superset of s itself. However, s^* clearly supports $B(n+1)$, and therefore also still supports $\exists x.B(x)$. This shows that any state supporting $\exists x.B(x)$ can be extended to a larger state which still supports $\exists x.B(x)$, and therefore no state supporting $\exists x.B(x)$ can be maximal.

This example shows that our notion of basic compliant responses, which makes crucial reference to maximal supporting states, does not always yield satisfactory results in the first-order setting. At first sight, it is tempting to conclude from this observation that there must be something wrong with the given notion of basic compliant responses. However, the problem is deeper than that. Namely, the following example, again from [2,3], shows that the very notion of meaning assumed in Inq_A is not fine-grained enough to serve as a basis for a suitable notion of compliance in the first-order setting.

Example 5 (The positive boundedness formula). Consider the following variant of the boundedness formula: $\exists x(x \neq 0 \wedge B(x))$. This formula says that there is a *positive* upper bound for P. Intuitively, it differs from the ordinary boundedness formula in that it does not license $B(0)$ as a compliant response. However, in terms of support, $\exists x(x \neq 0 \wedge B(x))$ and $\exists x.B(x)$ are equivalent. Thus, support is not fine-grained enough to capture the fact that these formulas intuitively do not have the same range of compliant responses.

3 An inquisitive witness semantics

In this section we will develop a first-order inquisitive witness semantics, Inq_W, which explicitly reflects the idea that an existentially quantified sentence like $\exists x.Px$ is supported in a state if and only if there is a specific witness in that state which is known to have the property P.

This idea is not entirely new. For instance, when informally describing the clause for existential quantification in Inq_A, Ciardelli [3] writes that "an existential will only be supported in those states where a specific witness for the existential is known." However, in Inq_A, states merely encode a certain body of information. To know a witness for a certain property is simply to know that the property holds of a specific individual. To say that a sentence *introduces a witness*, then, is just to say that the sentence provides the information that a certain individual has a certain property. But notice that, on this notion of witnesses, (i) a sentence may introduce infinitely many witnesses and (ii) these witnesses need not even be mentioned explicitly by the sentence. To deal with compliance, we need a stricter notion of witnesses: only individuals that are explicitly mentioned in the conversation should count as such. So, we need to devise a system which keeps track of the mentioned individuals, alongside the information that has been provided about them.

3.1 Witnesses, states, and support

In developing such a system, the first question to ask is what our formal notion of witnesses should be. The simplest answer would be that witnesses are objects in the domain D. This is indeed sufficient for the simplest cases of existential quantification. For instance, it would be reasonable to think of a state s as supporting a sentence $\exists x.Px$ just in case there is a specific object $d \in D$ which is known in s to have the property P. However, this notion of witnesses as objects in D is not general enough. In particular, it becomes problematic when we consider formulas where an existential quantifier is embedded under a universal quantifier. For instance, it would not be appropriate to think of a state s as supporting a sentence $\forall x.\exists y.Rxy$ just in case there is a specific object $d \in D$ which is known in s to stand in the relation R with all other objects in D. Intuitively, this is not what $\forall x.\exists y.Rxy$ requires.

To avoid problems of this sort, we will take witnesses to be *functions* from D^n to D, where $n \geq 0$. Notice that some of these functions are 0-place functions into D, which can simply be identified with objects in D. So witnesses *can* still be objects in D. But they can be other things as well.

In the definitions below, we will assume a fixed first-order language $\mathcal{L}$ and a fixed discourse-model $\mathbb{D} = \langle D, I \rangle$ for $\mathcal{L}$.

Definition 16 (Witnesses).

- *For any $n \in \mathbb{N}$, let $D^\star_n$ be the set of functions $\delta\colon D^n \to D$.*
- *Then $D^\star = \bigcup_{n \geq 0} D^\star_n$ is the set of all witnesses based on D.*

The next step is to reconsider our notion of a state. Before, states were sets of worlds, reflecting a certain body of information. Now states will not only reflect a certain body of information, but also contain a set of witnesses.

Definition 17 (States with witnesses).

- *A $\mathbb{D}$-state is a pair $\langle V, \Delta \rangle$, where V is a set of $\mathbb{D}$-models and Δ is a finite set of witnesses based on D, which contains the identity function $id : D \to D$.*
- *The set of all $\mathbb{D}$-states is denoted by $\mathcal{S}_{\mathbb{D}}$.*
- *If $s = \langle V, \Delta \rangle$ is a $\mathbb{D}$-state, then $\mathsf{worlds}(s) := V$ and $\mathsf{witn}(s) := \Delta$.*

We will often drop reference to $\mathbb{D}$, and simply refer to $\mathbb{D}$-states as states. The set of all states is partially ordered by the following *extension* relation.

Definition 18 (Extension). *Let s and t be two states. Then we say that s is an extension of t, $s \geq t$, iff $\mathsf{worlds}(s) \subseteq \mathsf{worlds}(t)$ and $\mathsf{witn}(t) \subseteq \mathsf{witn}(s)$.*

Notice that there is a minimal state, namely $\mathsf{top} := (W, \{id\})$, of which any other state is an extension. The extension relation will be used in the support definition, in particular in the clause for implication: a state s supports an implication iff every extension of s that supports the antecedent, supports the consequent as well.

Before turning to the definition of support, however, we introduce two more auxiliary notions. The first is the notion of a *witness feed*. The role of these witness feeds will be similar to that of assignments: they will be used to store certain information in evaluating whether or not a certain formula is supported by a certain state. In particular, they play a role in evaluating existentially quantified formulas in the scope of one or more universal quantifiers. This will be further explained once we have specified the support relation.

Definition 19 (Witness feeds). *A witness feed ε is a finite subset of D.*

Finally, we assume that the interpretation I of individual constants and function symbols in our discourse model $\mathbb{D}$ is extended in the following natural way to an interpretation of all terms $t \in \mathcal{L}$: if the free variables occurring in t are, orderly, $x_1, \ldots, x_n$, then $I(t)$ is the function $D^n \to D$ which maps a tuple $(d_1, \ldots, d_n) \in D^n$ to the element $d \in D$ denoted by the term t in $\mathbb{D}$ when x_i is interpreted as d_i for all $i = 1, \ldots, n$.

We now have all the necessary ingredients to state the support relation.

Definition 20 (Support in Inq_W).
Let s be a $\mathbb{D}$-state, g an assignment, ε a witness feed, and φ a formula in $\mathcal{L}$.

$$
\begin{array}{lll}
s \models_{g,\varepsilon} R(t_1, \ldots, t_n) & \textit{iff} & \textit{(i)}\ \mathsf{worlds}(s) \subseteq |R(t_1, \ldots, t_n)|_g \\
 & & \textit{(ii)}\ I(t_i) \in \mathsf{witn}(s)\ \textit{for}\ i = 1, \ldots, n \\
s \models_{g,\varepsilon} \bot & \textit{iff} & \mathsf{worlds}(s) = \emptyset \\
s \models_{g,\varepsilon} \varphi \wedge \psi & \textit{iff} & s \models_{g,\varepsilon} \varphi\ \textit{and}\ s \models_{g,\varepsilon} \psi \\
s \models_{g,\varepsilon} \varphi \vee \psi & \textit{iff} & s \models_{g,\varepsilon} \varphi\ \textit{or}\ s \models_{g,\varepsilon} \psi \\
s \models_{g,\varepsilon} \varphi \to \psi & \textit{iff} & \forall t \geq s : \textit{if}\ t \models_{g,\varepsilon} \varphi\ \textit{then}\ t \models_{g,\varepsilon} \psi \\
s \models_{g,\varepsilon} \forall x.\varphi & \textit{iff} & s \models_{g[x/d],\varepsilon \cup \{d\}} \varphi\ \textit{for all}\ d \in D \\
s \models_{g,\varepsilon} \exists x.\varphi & \textit{iff} & s \models_{g[x/\delta(e_1,\ldots,e_n)],\varepsilon} \varphi\ \textit{for some}\ \delta \in \mathsf{witn}(s)\ \textit{and}\ e_1, \ldots, e_n \in \varepsilon
\end{array}
$$

We will use $s \models_g \varphi$ as an abbreviation of $s \models_{g,\emptyset} \varphi$. The clauses that have changed w.r.t. Inq_A are those for atomic formulas, implication, universal quantification, and existential quantification. Let us look at these four clauses in some detail.

Atoms. For a state s to support an atomic sentence $R(t_1, \ldots, t_n)$, the sentence has to be true in all worlds in $\mathsf{worlds}(s)$, as before, but moreover, for every term t_i, the function $I(t_i)$ that it denotes must be available as a witness in $\mathsf{witn}(s)$. To illustrate this, consider the formula $R(a, f(b))$ where a and b are individual constants and f is a unary function symbol. Suppose $I(a) = d_1$ and $I(f(b)) = d_2$: then a state s supports the sentence $R(a, f(b))$ if and only if (i) for every $M \in \mathsf{worlds}(s)$ we have that $\langle d_1, d_2 \rangle \in M(R)$, and (ii) d_1 and d_2 are available as witnesses in $\mathsf{witn}(s)$.

Recall that in uttering a sentence, a speaker proposes to update the common ground of the conversation in such a way that it comes to support the sentence. Thus, in particular, in uttering $R(a, f(b))$, a speaker proposes to add d_1 and d_2 to the witness set of the common ground. In this sense, we can think of atomic sentences like $R(a, f(b))$ as introducing new witnesses. We will see that other sentences, in particular existentials, may request a response that introduces new witnesses.

Implication. In order to determine whether a state s supports an implication $\varphi \to \psi$ we have to consider all extensions t of s that support φ. An extension t of s is a state such that $\mathsf{worlds}(t) \subseteq \mathsf{worlds}(s)$ and $\mathsf{witn}(t) \supseteq \mathsf{witn}(s)$. Thus, it may be that all the extensions of s that support φ contain certain witnesses that are not contained in s itself. This means that if ψ requires certain witnesses, as long as we need to introduce them to support φ, it is not necessary for s as such to already contain them for the implication to be supported in s.

To illustrate this, let us show that $\mathsf{top} \models_{g,\varepsilon} Pa \rightarrow \exists x.Px$. Given the atomic clause, every $t \geq \mathsf{top}$ that supports Pa must be such that $I(a) \in \mathsf{witn}(t)$. In other words, every $t \geq \mathsf{top}$ that supports Pa contains a witness, namely $I(a)$, which is known to have the property P. It follows that $t \models_{g,\varepsilon} \exists x.P(x)$, which in turn means that $\mathsf{top} \models_{g,\varepsilon} Pa \rightarrow \exists x.Px$, even though top itself does not contain any witnesses besides the identity function.

Universal quantification. The clause for universal quantification is very much like the clause we had in Inq_A. Only now the witness feed plays a role as well. In determining whether a state s supports a formula $\forall x.\varphi$ we do not only set the current assignment g to $g[x/d]$, but we simultaneously augment the current witness feed ε with the same object d. Then we check whether φ is supported by s relative to the adapted assignment and the augmented witness feed. As we will see below, the augmented witness feed is put to use when φ contains an existential quantifier.

Existential quantification. In checking whether $s \models_{g,\varepsilon} \exists x.\varphi$ holds, we have to check whether $s \models_{g[x/d],\varepsilon} \varphi$ holds for some object $d \in D$ which is obtained by applying some witness $\delta \in \mathsf{witn}(s)$ to objects $e_1, \ldots, e_n$ in the witness feed. Thus, as desired, support of an existentially quantified sentence $\exists x.Px$ now really requires the presence of a witness which is known to have the property P. This means that in uttering $\exists x.Px$, a speaker requests a response that introduces a suitable witness and then establishes of this witness that it has the property P.

Example 6 (Interaction between existentials and universals). Consider the sentence $\forall x.\exists y.Rxy$. In order to determine whether $s \models_g \forall x.\exists y.Rxy$, we have to check whether $s \models_{g[x/d],\{d\}} \exists y.Rxy$ for all $d \in D$. And this means that we have to verify whether for every $d \in D$, there is a witness $f \in \mathsf{witn}(s)$ such that $s \models_{g[x/d][y/f(d,\ldots,d)],\{d\}} Rxy$. This witness f may be an element of the domain, a unary function, or a function of higher arity. It may also be the identity function, which means that the element d introduced by the universal can be used as a witness for the existential. This, then, is how universal and existential quantifiers interact: universal quantifiers add objects to the witness feed, and these objects then serve as the input for functional witnesses that may be needed for existentials in the scope of the universal. In this way, the witness that is required for the embedded existential in $\forall x.\exists y.Rxy$ may functionally depend on the value of x under the current assignment. We will return to this example in section 3.5, where we illustrate the potential relevance of Inq_W for natural language semantics.

As in Inq_A, support is *persistent*. That is, if a state s supports a formula φ relative to a certain assignment g and a certain witness feed ε, then any extension of s also supports φ relative to g and ε.

Fact 8 (Persistence) *If* $s \models_{g,\varepsilon} \varphi$ *and* $t \geq s$*, then* $t \models_{g,\varepsilon} \varphi$

Also as in Inq_A, we take $\neg\varphi$ to be an abbreviation of $\varphi \to \bot$, and $!\varphi$ an abbreviation of $\neg\neg\varphi$. The derived clauses for $\neg\varphi$ and $!\varphi$ read as follows.

Fact 9 (Support for negation)

- $s \models_{g,\varepsilon} \neg\varphi$ *iff for all* $M \in \mathsf{worlds}(s)$*:* $M \not\models_g \varphi$ *classically*
- $s \models_{g,\varepsilon} !\varphi$ *iff for all* $M \in \mathsf{worlds}(s)$*:* $M \models_g \varphi$ *classically*

3.2 Propositions, entailment, and equivalence

Based on the notion of support, we define the proposition expressed by a formula, and the notions of entailment and equivalence, just as in Inq_A. Recall that our definitions assume a fixed first-order language $\mathcal{L}$ and a fixed discourse-model $\mathbb{D} = \langle D, I \rangle$ for $\mathcal{L}$.

Definition 21 (Propositions, entailment, and equivalence).

1. $[\varphi]_g := \{s \in \mathcal{S}_{\mathbb{D}} \mid s \models_g \varphi\}$
2. $\varphi \models \psi$ *iff for all* s *and* g*: if* $s \models_g \varphi$*, then* $s \models_g \psi$
3. $\varphi \equiv \psi$ *iff* $\varphi \models \psi$ *and* $\psi \models \varphi$

In Inq_A, states were sets of possible worlds, ordered by inclusion, and we referred to *maximal* states supporting φ as *alternatives* for φ, where maximality was determined by the inclusion-order. Thus, alternatives for φ in Inq_A were *minimally informed* states supporting φ. In Inq_W, states are ordered by the extension relation, $\geq$, and alternatives for φ will be defined as $\geq$-minimal states supporting φ. Thus, in Inq_W alternatives for φ are states that support φ with a minimum amount of information and a minimal set of witnesses.

Definition 22 (Alternatives). *Let* φ *be a formula and* g *an assignment.*

1. *Every* $\geq$*-minimal element of* $[\varphi]_g$ *is called an* alternative *for* φ *relative to* g*.*
2. *The* alternative set *of* φ *relative to* g*,* $[\![\varphi]\!]_g$*, is the set of alternatives for* φ *relative to* g*.*

We also introduce notions of *factive* support, entailment, and equivalence, which ignore witness issues.

Definition 23 (Factive support, entailment, and equivalence).

1. $V \models^\star_g \varphi$ *iff there is a state* s *with* $\mathsf{worlds}(s) = V$ *such that* $s \models_g \varphi$
2. $\varphi \models^\star \psi$ *iff for all* V, g*: if* $V \models^\star_g \varphi$*, then* $V \models^\star_g \psi$
3. $\varphi \equiv^\star \psi$ *iff* $\varphi \models^\star \psi$ *and* $\psi \models^\star \varphi$

As long as we disregard witness issues, Inq_W coincides with Inq_A.

Fact 10 (Factive support and support in Inq_A)

$$V \models^\star_g \varphi \text{ in } \mathsf{Inq}_W \iff V \models_g \varphi \text{ in } \mathsf{Inq}_A$$

Clearly, this also means that factive entailment and equivalence in Inq_W amount to entailment and equivalence in Inq_A. We say that a formula is witness-insensitive in case it is supported by a state as soon as it is factively supported by the information available in that state.

Definition 24 (Witness insensitivity).
φ is witness insensitive iff for all s, g: if $\mathit{worlds}(s) \models^\star_g \varphi$, then $s \models_g \varphi$

Fact 11 (Partial characterization of witness insensitivity)

1. *An atomic formula is witness insensitive iff it does not contain any individual constant or a function symbol;*
2. *$\bot$ is witness insensitive;*
3. *If φ and ψ are witness insensitive, then $\varphi \vee \psi$ and $\varphi \wedge \psi$ are as well;*
4. *If ψ is witness insensitive, then $\varphi \to \psi$ is as well;*
5. *$\exists x.\varphi$ is not witness insensitive for any φ;*
6. *$\forall x.\varphi$ is witness insensitive iff φ is witness insensitive.*

Given that negation $\neg\varphi$ is defined as $\varphi \to \bot$, and non-inquisitive projection $!\varphi$ as $\neg\neg\varphi$, item 2 and 4 above guarantee that negation and non-inquisitive projection block witness sensitivity of their complement.

3.3 Informativeness and inquisitiveness

As before, we define the informative content of a sentence φ relative to an assignment g as the set of worlds that are contained in at least one state that supports φ relative to g.

Definition 25 (Informative content). $\mathit{info}_g(\varphi) := \bigcup\{\mathit{worlds}(s) \mid s \in [\varphi]_g\}$.

Also as before, the informative content of a sentence φ relative to an assignment g always coincides with the *truth set* of φ relative to g, $|\varphi|_g$, i.e., the set of worlds that satisfy φ in classical first-order logic relative to g. So as far as informative content is concerned, Inq_W does not diverge from classical first-order logic.

Fact 12 (Informative content is classical) *For any φ, g:* $\mathit{info}_g(\varphi) = |\varphi|_g$

In terms of the informative content of a formula, we define whether it is informative and/or inquisitive.

Definition 26 (Inquisitiveness and informativeness in a state).

- φ *is* informative in s w.r.t. g *iff* $\mathit{worlds}(s) \cap \mathit{info}_g(\varphi) \neq \mathit{worlds}(s)$
- φ *is* inquisitive in s w.r.t. g *iff* $\mathit{worlds}(s) \cap \mathit{info}_g(\varphi) \not\models^\star_g \varphi$

Definition 27 (Absolute inquisitiveness and informativeness).

- φ *is* informative *iff for some g:* $\mathit{info}_g(\varphi) \neq W$
- φ *is* inquisitive *iff for some g:* $\mathit{info}_g(\varphi) \not\models^\star_g \varphi$

Fact 13 (Informativeness and inquisitiveness in Inq_W and Inq_A)

- *φ is informative in Inq_W iff φ is informative in Inq_A*
- *φ is inquisitive in Inq_W iff φ is inquisitive in Inq_A*

All notions in Inq_A that are defined in terms of informativeness and inquisitiveness, such as the notions of assertions, questions, and hybrids, remain precisely the same in intension and extension. In particular:

Fact 14 *For any φ, $!\varphi$ is an assertion and $?\varphi$ is a question.*

However, among assertions and questions there is a further distinction now between witness sensitive and witness insensitive ones.

3.4 The boundedness problem resolved

Now that we have discussed some of the basic logical properties of Inq_W, let us return to the problem that we set out to resolve. The crucial problem was that the boundedness formulas were semantically indistinguishable in Inq_A. They were supported by exactly the same states. As a result, it was impossible in Inq_A to capture the intuition that these formulas have a different range of compliant responses. This problem no longer arises in Inq_W.

Fact 15 (The boundedness formulas) *The boundedness formula and the positive boundedness formula are not equivalent in Inq_W.*

Proof. Consider a state s such that:

- $\mathsf{worlds}(s) = \{M\}$, where $M(P) = \{0\}$
- $\mathsf{witn}(s) = \{0\}$

This state factively supports both $\exists x.B(x)$ and $\exists x.(x > 0 \wedge B(x))$. However, while the boundedness formula is supported in s *tout court*, $s \models \exists x.B(x)$, the positive boundedness formula is not, $s \not\models \exists x.(x > 0 \wedge B(x))$. So, the boundedness formula and the positive boundedness formula are not equivalent in Inq_W (although they *are* factively equivalent, and therefore equivalent in Inq_A). □

The notion of basic compliant responses that we had in Inq_A carries over straightforwardly to Inq_W. Recall that in Inq_A, the basic compliant responses to a sentence φ were intuitively characterized as those responses that provide precisely enough information to establish a state that supports φ. In Inq_W, states do not only contain information but also witnesses, and support sometimes requires the presence of such witnesses. Thus, in Inq_W the basic compliant responses to a sentence φ are intuitively characterized as those responses that provide precisely enough information and precisely enough witnesses to establish a state that supports φ. This intuition is formalized exactly as it was in Inq_A.

Definition 28 (Basic compliant responses).
ψ is a basic compliant response to φ iff $[\![\psi]\!] = \{\alpha\}$ for some $\alpha \in [\![\varphi]\!]$.

Fact 16 (Basic compliant responses to the boundedness formulas)

- *For any* $n \geq 0$*,* $B(n)$ *is a basic compliant response to* $\exists x.Bx$
- *For any* $n > 0$*,* $B(n)$ *is a basic compliant response to* $\exists x.(x \neq 0 \wedge Bx)$*, but* $B(0)$ *is not a basic compliant response to* $\exists x.(x \neq 0 \wedge Bx)$*.*

Finally, we note that one basic compliant response may intuitively be preferred over another. For instance, $B(1)$ and $B(135)$ are both basic compliant responses to $\exists x.Bx$. However, $B(1)$ is intuitively preferred over $B(135)$. If the information state of the responder supports $B(1)$ then it would be misleading for her to actually choose $B(135)$ as a response. In general, if ψ and χ are two basic compliant responses to φ, and ψ factively entails χ, then ψ is preferred over χ as a response to φ.[4]

Definition 29 (Comparing basic compliant responses).
Let φ *be an inquisitive initiative, let* ψ *and* χ *be two basic compliant responses to* φ*, and let* σ *be an information state, i.e., a set of worlds. Then:*

1. ψ *is* preferred *over* χ *as a response to* φ *iff* $\psi \models^\star \chi$ *and* $\chi \not\models^\star \psi$*.*
2. ψ *is an* optimal response *to* φ *in* σ *iff*
   - $\sigma \subseteq \mathit{info}(\psi)$*, and*
   - *for every basic compliant response* ξ *to* φ *that is preferred over* ψ*,* $\sigma \not\subseteq \mathit{info}(\xi)$*.*

To illustrate the notion of an optimal response, consider an information state consisting of three worlds, one where the highest element of P is 5, one where it is 14, and one where it is 3. The optimal response to $\exists x.Bx$ in this information state is $B(14)$. This accounts for the intuition that, on the one hand, any response $B(n)$ with $n < 14$, even though compliant, would be *qualitatively* inappropriate, while any response $B(n)$ with $n > 14$ would be *quantitively* dispreferred. The only optimal response in this scenario is $B(14)$.

3.5 Relevance for natural language semantics

Our discussion so far has focussed on a particular foundational issue in inquisitive semantics. We will end with a brief discussion and illustration of the potential relevance of the resulting system for natural language semantics. Inquisitive semantics is primarily intended to offer a logical *framework* in which different theories about informative and inquisitive constructions in natural language can be formulated and compared. For instance, Hamblin's classical account of interrogatives [8] can be formulated in terms of inquisitive existential quantification ($\exists x.Px$), while Groenendijk and Stokhof's partition account [7] is covered

[4] Notice that one basic compliant response never strictly entails another basic compliant response *tout court*. However, it may very well be the case that one basic compliant response *factively* entails another. In Inq_A the distinction between entailment and factive entailment does not exist, and basic compliant responses are always incomparable, i.e., one never strictly entails another.

by universal quantification over polar questions ($\forall x.?Px$). Unlike the partition framework, inquisitive semantics also allows for a straightforward analysis of conditional questions (represented as $p \rightarrow ?q$), and Velissaratou's account of *which*-questions [10] can be formulated in terms of universal quantification over such conditional questions ($\forall x(Px \rightarrow ?Qx)$). Finally, Inq_W seems particularly suitable for the analysis of questions with quantifiers like *Who does every man like?*. As has been discussed widely in the literature (e.g., [1,5,7]), such questions allow for different types of responses, e.g., *Mary*, *himself*, or *his mother*. If this question is formally represented as $\forall x.\exists y.Rxy$, these different types of responses are accounted for in a straightforward and uniform way.

To see this, consider what is needed for a state s to support $\forall x.\exists y.Rxy$. If we assume that $\mathsf{witn}(s)$ does not contain any witnesses, apart from the identity function, which is always an element of $\mathsf{witn}(s)$, then we must have that $\langle d, d\rangle \in M(R)$ for every $d \in D$ and every $M \in \mathsf{worlds}(s)$. The $\geq$-minimal state that satisfies this condition is one of the alternatives for $\forall x.\exists y.Rxy$. It is also the unique alternative for the response *himself* ($\forall x.Rxx$). Thus, this is a basic compliant response.

Now consider a state s such that $\mathsf{witn}(s)$ contains an object m, and such that $\langle d, m\rangle \in M(R)$ for every $d \in D$ and every $M \in \mathsf{worlds}(s)$. The $\geq$-minimal state that satisfies these conditions is another alternative for $\forall x.\exists y.Rxy$. It is also the unique alternative for the response *Mary* ($\forall x.Rxm$). Thus, this is another basic compliant response.

Finally, consider a state s such that $\mathsf{witn}(s)$ contains a 1-place function f which maps every individual in D to his mother, and such that $\langle d, f(d)\rangle \in M(R)$ for every $d \in D$ and every $M \in \mathsf{worlds}(s)$. The $\geq$-minimal state that satisfies these conditions is again one of the alternatives for $\forall x.\exists y.Rxy$. It is also the unique alternative for the response *his mother* ($\forall x.R(x, f(x))$). Thus, this is yet another basic compliant response.

4 Conclusions and an open question

The goal we have been pursuing in this paper was to provide a notion of meaning that does not only embody the informative and inquisitive content of a sentence, but also determines the range of compliant responses to that sentence. We focused on basic compliant responses, which were intuitively characterized as responses that resolve a given issue without providing more information than necessary. Inquisitive semantics is intended, among other things, to provide a semantic framework in which this intuitive notion can be suitably formalized.

Inq_A fulfills this purpose in a simple and perspicuous way as long as we restrict our attention to the language of propositional logic. In this setting, the basic compliant responses to a sentence correspond exactly to the *alternatives* for that sentence, i.e., the maximal states supporting the sentence. Unfortunately, this simple picture breaks down in the first-order setting. It turns out that the notion of meaning adopted in Inq_A is not fine-grained enough to capture subtle differences in the range of compliant responses to certain first-order sentences.

To tackle this problem we developed a more fine-grained semantics, Inq_W, which is based on a slightly more involved notion of states than the one assumed in Inq_A. This refinement allowed us to distinguish sentences which have the same informative and inquisitive content but a different range of compliant responses.

We saw that the two boundedness formulas, which were problematic for Inq_A, are suitably dealt with in Inq_W. However, this does of course not necessarilly mean that Inq_W allows us to suitably deal with compliance in general. We cannot exclude that there are further examples, not considered so far, that are problematic for Inq_W.[5] In this light, it is of interest to identify certain general requirements that Inq_W should satisfy in order to be considered an adequate system. One natural requirement is that the semantics should allow for at least one basic compliant response to every sentence. That is, every sentence should have at least one $\geq$-minimal supporting state.

Requirement 1 (Existence of $\geq$-minimal supporting states).
For any φ and every g, $[\varphi]_g$ should have at least one $\geq$-minimal element.

The analogue of this requirement does not hold for Inq_A, as illustrated by the boundedness formula. Whether it holds for Inq_W is still an open question. A positive answer to this question would consolidate Inq_W as a natural setting for the analysis of information exchange through conversation. A negative answer, on the other hand, would seem to indicate that Inq_W can only be seen as an intermediate step towards a satisfactory system of inquisitive semantics.

[5] In fact, just before submitting this paper, we became aware of an example which indeed seems problematic. Consider the setting of the earlier boundedness examples, but now suppose that our language has two unary predicates, P and Q, rather than just P. Let $B_P(x) \coloneqq \forall y(P(y) \rightarrow y \leq x)$ and let $B_Q(x) \coloneqq \forall y(Q(y) \rightarrow y \leq x)$, so that $\exists x B_P(x)$ and $\exists x B_Q(x)$ are two distinct boundedness formulas.

Now consider $\exists x B_P(x) \wedge \exists x B_Q(x)$. Since $B_P(3)$ is a basic compliant response to $\exists x B_P(x)$ and $B_Q(4)$ is a basic compliant response to $\exists x B_Q(x)$, we would expect $B_P(3) \wedge B_Q(4)$ to come out as a basic compliant response to the conjunction. However, $B_P(3) \wedge B_Q(4)$ strictly entails $B_P(4) \wedge B_Q(4)$ which is issue-resolving, so that the system does not qualify $B_P(3) \wedge B_Q(4)$ as a basic compliant response. Intuitively, this is unexpected: in fact $B_P(3) \wedge B_Q(4)$ would seem preferable over $B_P(4) \wedge B_Q(4)$ as a response to $\exists x B_P(x) \wedge \exists x B_Q(x)$.

This example does not immediately seem to show, however, that there is something fundamentally wrong with Inq_W. Rather, it seems to indicate that the notion of basic compliant responses that we have adopted sometimes gives unexpected results. After all, $B_P(3) \wedge B_Q(4)$ does introduce more witnesses than is needed to resolve the issue raised by $\exists x B_P(x) \wedge \exists x B_Q(x)$. In that sense, it is to be expected that it does not qualify as a basic compliant response. On the other hand, there is an intuitive understanding of the notion of compliance under which we do expect $B_P(3) \wedge B_Q(4)$ to count as a compliant response. It may be possible to capture this intuition within Inq_W, with a different formal notion of compliance than the one adopted in this paper. This issue needs to be addressed in more detail in future work.

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