An inquisitive semantics with witnesses*

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October 31, 2011

Abstract

A central notion in inquisitive semantics is that of *support*. A common way to formulate the semantics is to start with a recursive definition of when a state supports a sentence, and then define the proposition expressed by a sentence as the set of all (maximal) states supporting that sentence. This approach is similar to the one that is usually taken in classical logic. There we start with a recursive definition of truth, and then define the proposition expressed by a sentence as the set of all worlds where the sentence is true. Thus, the role of support in inquisitive semanics is comparable to that of truth in classical logic.

However, Ciardelli (2009, 2010) has shown that there are certain sentences in the language of first-order logic, the so-called *boundedness formulas*, which are equivalent in terms of support, even though, intuitively, they license a different range of responses. Ciardelli concludes from this observation that a support-based inquisitive semantics is not fine-grained enough in the first-order setting.

In the semantics we will propose in this paper, states do not only contain information, but also a set of *witnesses*. The main feature of the semantics, then, is that an existentially quantified sentence like $\exists x.Px$ is only supported in a state if there is a specific witness in that state which is known to have the property P. As a result, an inquisitive sentence may not only request a response that provides certain information, but also a response that introduces a certain witness. Thus, the notion of inquisitiveness is richer than in the basic first-order system. Because the notion of inquisitiveness is enriched in this way, the semantics is able to make more fine-grained distinctions. In particular, it suitably assigns different semantic values to the boundedness formulas. At the same time, unlike the semantics that Ciardelli proposed to avoid the boundedness problem, the semantics developed here is still support-based.

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*This is work in progress. The paper has been presented in this form at the *Prague Workshop on the Logic of Questions*, October 26, 2011.

1 Setting the stage

The main purpose of inquisitive semantics is to enrich the notion of semantic meaning in such a way that it does not only embody informative content, but also inquisitive content. Of course, there are different ways to achieve this goal. The starting point of the present paper is what we currently see as the most basic implementation of inquisitive semantics, which we will refer to as INQB. This system was initially specified for the language of propositional logic (Ciardelli, 2008; Groenendijk and Roelofsen, 2009). The system has been motivated quite extensively, and its logical properties are well-understood (Ciardelli and Roelofsen, 2011; Roelofsen, 2011). However, the issues that arise in extending INQB to the first-order setting are not yet fully resolved.

In section 1.1, we will briefly review the definition of INQB for the language of propositional logic. Subsequently, in section 1.2, we will consider the main issues that arise in extending INQB to the first-order setting, as pointed out by Ciardelli (2009, 2010). This will set the stage for our inquisitive witness semantics, to be introduced in section 2.

1.1 A semantics for the language of propositional logic

We start with a brief recapitulation of INQB for the language of propositional logic, as specified in Ciardelli (2009); Groenendijk and Roelofsen (2009); Ciardelli and Roelofsen (2011).

Definition 1 (Language)**.** Let $\mathcal{P}$ be a finite set of proposition letters. We denote by $\mathcal{L}_\mathcal{P}$ the set of formulas built up from letters in $\mathcal{P}$ and $\bot$ using the binary connectives $\wedge, \vee$ and $\to$. We will refer to $\mathcal{L}_\mathcal{P}$ as the propositional language based on $\mathcal{P}$.

We will also make use of the following abbreviations: $\neg\varphi$ for $\varphi \to \bot$, $!\varphi$ for $\neg\neg\varphi$, and $?\varphi$ for $\varphi \vee \neg\varphi$.

Definition 2 (Worlds)**.**
A *world* is a function from $\mathcal{P}$ to $\{0, 1\}$. We denote by W the set of all worlds.

Definition 3 (States)**.**
A *state* is a set of worlds. We denote by $\mathcal{S}$ the set of all states.

Definition 4 (Support)**.**

1. $s \models p$ iff $\forall w \in s : w(p) = 1$
2. $s \models \bot$ iff $s = \emptyset$
3. $s \models \varphi \wedge \psi$ iff $s \models \varphi$ and $s \models \psi$
4. $s \models \varphi \vee \psi$ iff $s \models \varphi$ or $s \models \psi$
5. $s \models \varphi \to \psi$ iff $\forall t \subseteq s$: if $t \models \varphi$ then $t \models \psi$

It follows from the above definition that the empty state supports any formula φ. Thus, we may think of $\emptyset$ as the *absurd* state.

Fact 1 (Persistence)**.** If $s \models \varphi$ then for every $t \subseteq s$: $t \models \varphi$

Fact 2 (Singleton states behave classically)**.** For any world w and formula φ:

$$\{w\} \models \varphi \quad \Longleftrightarrow \quad w \models \varphi \text{ in classical propositional logic}$$

It can be derived from definition 17 that the support-conditions for $\neg\varphi$ and $!\varphi$ are as follows.

Fact 3 (Support for negation)**.**

1. $s \models \neg\varphi \quad \text{iff} \quad \forall w \in s : w \not\models \varphi$
2. $s \models\, !\varphi \quad \text{iff} \quad \forall w \in s : w \models \varphi$

In terms of support, we define the *possibilities* for a sentence φ and the *proposition* expressed by φ. We also define the *truth-set* of φ, which is the meaning that would be associated with φ in a classical setting.

Definition 5 (Truth sets, possibilities, propositions)**.** Let φ be a formula.

1. A *possibility* for φ is a maximal state supporting φ, that is, a state that supports φ and is not properly included in any other state supporting φ.
2. The *proposition* expressed by φ, $[\varphi]$, is the set of possibilities for φ.
3. The *truth set* of φ, $|\varphi|$, is the set of all worlds w such that $\{w\} \models \varphi$.

The following result guarantees that the proposition expressed by a formula completely determines which states support that formula, and vice versa.

Fact 4 (Support and possibilities)**.** For any state s and any formula φ:

$$s \models \varphi \quad \Longleftrightarrow \quad s \text{ is contained in a possibility for } \varphi$$

Example 1 (Disjunction)**.** Inquisitive semantics differs from classical semantics in its treatment of disjunction. To see this, consider figures 1(a) and 1(b). In these figures, it is assumed that $\mathcal{P} = \{p, q\}$; world 11 makes both p and q true, world 10 makes p true and q false, etcetera. Figure 1(a) depicts the truth set—that is, the classical meaning—of $p \vee q$: the set of all worlds that make p, q or both p and q true. Figure 1(b) depicts the proposition expressed by $p \vee q$ in inquisitive semantics. It consists of two possibilities. One possibility is made up of all worlds that make p true, and the other of all worlds that make q true.

We think of a sentence φ as expressing a proposal to update the common ground of a conversation—formally conceived of as a set of possible worlds—in such a way that the new common ground supports φ. In other words, given fact 4, a sentence proposes to update the common ground in such a way that the resulting common ground is contained in one of the possibilities for φ.

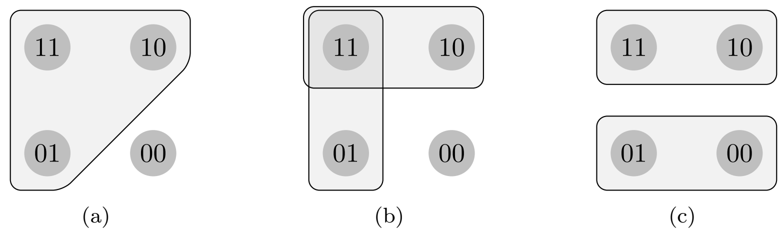

Figure 1: (a) classical picture of $p \vee q$, (b) inquisitive picture of $p \vee q$, and (c) polar question $?p$.

Worlds that are not contained in any state supporting φ will not survive any of the updates proposed by φ. In other words, if any of the updates proposed by φ is executed, all worlds that are not contained in any state supporting φ will be eliminated. Given fact 4, a world is contained in a state supporting φ iff it is contained in the union of all the possibilities for φ, $\bigcup[\varphi]$. Therefore, we refer to $\bigcup[\varphi]$ as the *informative content* of φ.

Definition 6 (Informative content)**.**

- $\mathsf{info}(\varphi) = \bigcup[\varphi]$

In the classical setting, the informative content of φ is captured by $|\varphi|$. The following result says that, as far as informative content goes, INQB does not diverge from classical propositional logic. In this sense, INQB is a conservative extension of classical propositional logic.

Fact 5. For any formula φ: $\mathsf{info}(\varphi) = |\varphi|$

A sentence φ is informative in a state s iff it proposes to eliminate at least one world in s, i.e., iff $s \cap \mathsf{info}(\varphi) \neq s$. On the other hand, φ is inquisitive in s iff in order to reach a state $s' \subseteq s$ that supports φ it is not enough to incorporate the informative content of φ itself into s, i.e., $s \cap \mathsf{info}(\varphi) \not\models \varphi$, which means that φ requests a response from other participants that provides additional information.

Definition 7 (Inquisitiveness and informativeness in a state)**.**

- φ is *informative* in s iff $s \cap \mathsf{info}(\varphi) \neq s$
- φ is *inquisitive* in s iff $s \cap \mathsf{info}(\varphi) \not\models \varphi$

Besides these notions of informativeness and inquisitiveness *relative to a state* we may also define absolute notions of informativeness and inquisitiveness.

Definition 8 (Absolute inquisitiveness and informativeness)**.**

- φ is *informative* iff it is informative in W, i.e., iff $\mathsf{info}(\varphi) \neq W$

- φ is *inquisitive* iff it is inquisitive in W, i.e., iff $\mathsf{info}(\varphi) \not\models \varphi$

Inquisitive sentences are characterized here in terms of support and informative content. Alternatively, they may also be characterized in terms of the number of possibilitites in the proposition that they express.

Fact 6 (Alternative characterization of inquisitiveness)**.**

- φ is *inquisitive* iff $[\varphi]$ contains at least two possibilities.

Example 2 (Disjunction continued)**.** As in the classical setting, $p \vee q$ is *informative*, in that it proposes to eliminate the world where both p and q are false. But it is also *inquisitive*, in that it proposes to move to a state that supports p or to a state that supports q, while merely eliminating the world where both p and q are false is not sufficient to reach such a state. Thus, $p \vee q$ requests a response that provides additional information. This inquisitive aspect of meaning is not captured in the classical setting.

Definition 9 (Questions, assertions, and hybrids)**.**

- φ is a *question* iff it is not informative;
- φ is an *assertion* iff it is not inquisitive;
- φ is a *hybrid* iff it is both informative and inquisitive.

Example 3 (Questions, assertions, and hybrids)**.** We saw above that $p \vee q$ is both informative and inquisitive, i.e., hybrid. The proposition depicted in figure 1(a) is the proposition expressed by $!(p \vee q)$. This proposition consists of exactly one possibility. So $!(p \vee q)$ is an assertion. The proposition depicted in figure 1(c) is expressed by $?p$. This proposition covers the entire logical space, so $?p$ does not propose to eliminate any world. That is, $?p$ is a question.[1]

1.2 Moving to the first-order setting

Now let us extend the propositional system specified above to the first-order setting, and describe the problem that arises in doing so. Our exposition here will closely follow that of Ciardelli (2010).

Let $\mathcal{L}$ be a first-order language. A *state* will now be a set of first-order models for $\mathcal{L}$. For simplicity, we will assume that all models in a state share the same domain and the same interpretation of individual constants and function symbols. Thus, every model in a state is based on a common *proto-model* $\mathbb{D} = \langle D, I \rangle$, where D is a domain and I an interpretation of all individual

[1] Notice that questions, as defined here, are not necessarily inquisitive, and assertions are not necessarily informative. For instance, the tautology $!(p \vee \neg p)$ is both a question and an assertion, even though (or rather *because*) it is neither inquisitive nor informative. It is possible to give a slightly more involved definition of questions and assertions, which makes sure that the two notions are strictly disjoint (see Groenendijk and Roelofsen, 2009). This may be more desirable from a linguistic point of view, but the additional complexity is not quite relevant in the present setting, and is therefore avoided.

constants and function symbols in $\mathcal{L}$. If $\mathbb{D} = \langle D, I \rangle$ is a proto-model, and $M = \langle D_M, I_M \rangle$ a model such that $D_M = D$ and I_M coincides with I as far as individual constants and function symbols are concerned, then M is called a model based on $\mathbb{D}$, or simply a $\mathbb{D}$-model.

Definition 10 (States)**.**
A state is a set of models which are all based on the same proto-model.

For any assignment g, let $|\varphi|_g$ denote the *truth set* of φ relative to g, i.e., the set of models M such that $M, g \models \varphi$ in classical first-order logic. The most straightforward definition of support in the first-order setting is as follows.

Definition 11 (First-order support)**.**

1. $s, g \models \varphi \quad \text{iff} \quad \forall M \in s : M, g \models \varphi \qquad \text{for atomic } \varphi$
2. $s, g \models \bot \quad \text{iff} \quad s = \emptyset$
3. $s, g \models \varphi \wedge \psi \quad \text{iff} \quad s, g \models \varphi \text{ and } s, g \models \psi$
4. $s, g \models \varphi \vee \psi \quad \text{iff} \quad s, g \models \varphi \text{ or } s, g \models \psi$
5. $s, g \models \varphi \to \psi \quad \text{iff} \quad \forall t \subseteq s : \text{if } t, g \models \varphi \text{ then } t, g \models \psi$
6. $s, g \models \forall x.\varphi \quad \text{iff} \quad s, g[x/d] \models \varphi \text{ for all } d \in D$
7. $s, g \models \exists x.\varphi \quad \text{iff} \quad s, g[x/d] \models \varphi \text{ for some } d \in D$

Recall that in the propositional setting, we defined the *proposition* expressed by a sentence as the set of maximal states supporting the sentence. Ciardelli observes that this definition is problematic in the first-order setting, because now there are sentences that do not have any maximal supporting states.

Example 4 (The boundedness formula)**.** Consider a language which has a unary predicate symbol P, a binary function symbol $+$, and the set $\mathbb{N}$ of natural numbers as its individual constants. Consider the proto-model $\mathbb{D} = \langle D, I \rangle$, where $D = \mathbb{N}$, I maps every $n \in \mathbb{N}$ to itself, and $+$ is interpreted as addition. Let $x \leq y$ abbreviate $\exists z(x + z = y)$, let $B(x)$ abbreviate $\forall y(P(y) \to y \leq x)$, and for every $n \in \mathbb{N}$, let $B(n)$ abbreviate $\forall y(P(y) \to y \leq n)$. Intuitively, $B(n)$ says that n is greater than or equal to any number in P. In other words, $B(n)$ says that n is an *upper bound* for P.

A state s supports a formula $B(n)$, for some $n \in \mathbb{N}$, if and only if $B(n)$ is true in every model in s, that is, if and only if n is an upper bound for P in every M in s. Now consider the formula $\exists x.B(x)$, which intuitively says that there is an upper bound for P. This formula, which Ciardelli refers to as the *boundedness formula*, does not have a maximal supporting state. To see this, let s be an arbitrary state supporting $\exists x.B(x)$. Then there must be a number $n \in \mathbb{N}$ such that s supports $B(n)$, i.e., $B(n)$ must be true in all models in s. Now let M^* be the model in which P denotes the singleton set $\{n + 1\}$. Then M^* cannot be in s, because it does not make $B(n)$ true. Thus, the state s^* which is obtained from s by adding M^* to it is a proper superset of s itself. However, s^* clearly supports $B(n+1)$, and therefore also still supports $\exists x.B(x)$. This shows that any state supporting $\exists x.B(x)$ can be extended to a larger state which still supports $\exists x.B(x)$, and therefore no state supporting $\exists x.B(x)$ can be maximal.

Ciardelli concludes from this observation that propositions cannot be defined as sets of maximal supporting states in the first-order setting. However, his argument does not stop here. Based on an additional example, reproduced below, he argues that there is in fact no satisfactory way to define the meaning of first-order formulas in terms of support at all.

Example 5 (The positive boundedness formula)**.** Consider the following variant of the boundedness formula: $\exists x(x \neq 0 \wedge B(x))$. This formula says that there is a *positive* upper bound for P. Intuitively, it differs from the ordinary boundedness formula in that it does not license the response "Yes, zero is an upper bound for P." However, in terms of support, $\exists x(x \neq 0 \wedge B(x))$ and $\exists x.B(x)$ are equivalent. Thus, support is not fine-grained enough to capture the fact that these formulas intuitively do not license the same range of responses.

1.3 The maximality problem revisited

Ciardelli concludes from the maximality problem that there is no satisfactory way whatsoever to define the meaning of first-order formulas in terms of support. We would like to show that it is in fact possible to devise a support-based first order system that deals appropriately with the boundedness formulas. Before introducing this system, however, we will briefly revisit the maximality problem as presented by Ciardelli. In particular, it is very important to understand precisely what Ciardelli's requirements are for a "satisfactory way to define the meaning of first-order formulas."

Recall Ciardelli's first observation: as exemplified by the boundedness formula, certain intuitively non-contradictory sentences do not have maximal supporting states. This means that we cannot define the proposition expressed by a sentence as the set of maximal states supporting that sentence, as we did in the propositional setting. But this does not yet mean that there is no suitable way of defining propositions in terms of support. For instance, it would be very natural to define the proposition expressed by a sentence as the set of *all* states supporting that sentence, rather than just the maximal ones. This would be in direct analogy with the classical definition of the proposition expressed by a sentence as the set of all worlds that make that sentence true.

Proceeding in this way indeed gives rise to a well-behaved system, which has been arrived at via independent considerations as well (Roelofsen, 2011). In this system the informative content of a sentence φ can still be defined as the union of all the states in $[\varphi]$. Moreover, we can still define when a sentence is informative and when it is inquisitive in a state s, exactly as we did before: φ is informative in s iff $s \cap \mathsf{info}(\varphi) \neq s$, and φ is inquisitive in s iff $s \cap \mathsf{info}(\varphi) \not\models \varphi$.

Clearly, the maximality problem does not arise in this system, because propositions are not defined in terms of maximal supporting states any more.

In particular, the boundedness formula is associated with a proposition which suitably captures the information that this formula provides and the information that it requests. Namely, it provides the information that there is an upper bound for P, and it requests a response that provides enough information to establish for a particular number $n \in \mathbb{N}$, that it is an upper bound for P. So if these are the relevant requirements for a "satisfactory way to define the meaning of first-order formulas," then Ciardelli's conclusion is clearly too strong.

However, the second example, the one involving the *positive* boundedness formula, makes clear that Ciardelli's requirements are more demanding: he wants the proposition expressed by a sentence to capture not only the information that the sentence provides and the information that the sentence requests, but also the range of responses that the sentence "licenses." After all, the two boundedness formulas do not differ in the information that they provide or in the information that they request, they only differ in the range of responses that they license. And this, Ciardelli concludes, can never be captured if the meaning of first-order formulas is defined in terms of support.

From this closer examination of the maximality problem we draw the following three conclusions. First, referring to the problem at hand as the *maximality* problem may be a bit misleading, since the maximality issue itself is rather innocent and easily resolved. Second, as long as we only require of a proposition that it captures the information that a sentence provides and the information that a sentence requests, then it is appropriate to define the proposition expressed by a sentence as the set of all states supporting that sentence. And third, the real challenge posed by Ciardelli is to find a notion of meaning that captures differences in *licensing*, and that allows us in particular to account for the fact that the two boundedness formulas differ in this respect.

This challenge will be addressed in the next section. In the semantics we will develop, states do not only contain information, but also a set of *witnesses*. The main feature of the semantics, then, is that an existentially quantified sentence like $\exists x.Px$ is only supported in a state if there is a specific witness in that state which is known to have the property P. As a result, an inquisitive sentence may not only request a response that provides certain information, but also a response that introduces a certain witness. Thus, the notion of inquisitiveness is richer than in INQB. Because the notion of inquisitiveness is enriched in this way, the semantics is able to make more fine-grained distinctions. In particular, it suitably assigns different semantic values to the boundedness formulas. At the same time, unlike the semantics that Ciardelli proposed to avoid the boundedness problem, the semantics developed here will still be based on the notion of support.

2 An inquisitive witness semantics

In this section we will develop a first-order inquisitive witness semantics, INQW. This system will explicitly reflect the idea that an existentially quantified sentence like $\exists x.Px$ is supported in a state if and only if there is a specific witness in that state which is known to have the property P. This idea is not new. For instance, when informally describing the clause for existential quantification in INQB, Ciardelli (2010) writes that "an existential will only be supported in those states where a specific witness for the existential is known." And the idea has always been part of our own thinking about first-order inquisitive semantics as well. However, it has always remained at the level of informal intuitions, and was never fully incorporated into a concrete first-order system. This is exactly what we intend to do below.

2.1 Witnesses and states

The first question to ask is, of course, what our formal notion of *witnesses* should be. The simplest answer would be that witnesses are simply objects in the domain D. This is indeed sufficient for the simplest cases of existential quantification. For instance, it would be reasonable to think of a state s as supporting a sentence $\exists x.Px$ just in case there is a specific object $d \in D$ which is known in s to have the property P. However, this notion of witnesses as objects in D is not general enough. In particular, it becomes problematic when we consider formulas where an existential quantifier is embedded under a universal quantifier. For instance, it would not be appropriate to think of a state s as supporting a sentence $\forall x.\exists y.Rxy$ just in case there is a specific object $d \in D$ which is known in s to stand in the relation R with all other objects in D. Intuitively, this is not what $\forall x.\exists y.Rxy$ requires.

To avoid problems of this sort, we will take witnesses to be *functions* from D^n to D, where $n \geq 0$.[2] Notice that some of these functions are 0-place functions into D, which can be simply identified with objects in D. So witnesses *can* still be objects in D. But they can be other things as well.

In the definitions below, we will assume a fixed first-order language $\mathcal{L}$ and a fixed proto-model $\mathbb{D} = \langle D, I \rangle$ for $\mathcal{L}$.

Definition 12 (Witnesses)**.**

- For any $n \in \mathbb{N}$, let $D^\star{}_n$ be the set of functions $\delta \colon D^n \to D$.
- Then $D^\star = \bigcup_{n \geq 0} D^\star{}_n$ is the set of all witnesses based on D.

The next step is to reconsider our notion of a state. Before, states were sets of worlds, reflecting a certain body of information. Now states will not only reflect a certain body of information, but also contain a set of witnesses.

[2] Note that Skolem-functions are of this nature. And the Skolemnization of the sentence $\forall x.\exists y.Rxy$, a technique that is often used in proof systems for first order logic, is to transform $\forall x.\exists y.Rxy$ into $\exists f.\forall x.R(x, f(x))$. Note, the latter is not a sentence of first order logic, since it does not quantify over objects but over functions from objects to objects. But the idea is that we can prove $\forall x.\exists y.Rxy$ if we can prove that $\forall x.R(x, f(x))$ for some function symbol f. Just like that we can prove $\exists x.Px$ if we can prove Pc for some individual constant, i.e., some 0-place function expression c.

Definition 13 (States)**.**

- A $\mathbb{D}$-state is a pair $\langle V, \Delta \rangle$, where V is a set of $\mathbb{D}$-models and $\Delta \subseteq D^\star$.
- The set of all $\mathbb{D}$-states is denoted by $\mathcal{S}_\mathbb{D}$.
- If $s = \langle V, \Delta \rangle$ is a $\mathbb{D}$-state, then:
  - $\mathsf{worlds}(s) \coloneqq V$
  - $\mathsf{witn}(s) \coloneqq \Delta$
  - $s^\star \coloneqq \langle V, D^\star \rangle$
  - $s^\emptyset \coloneqq \langle \emptyset, \Delta \rangle$

In what follows, we will usually drop reference to $\mathbb{D}$, and simply refer to a $\mathbb{D}$-state as a state. The set of states is partially ordered by the following *extension* relation.

Definition 14 (Extension)**.** Let s and t be two states. Then we say that s is an *extension* of t, $s \geq t$, iff $\mathsf{worlds}(s) \subseteq \mathsf{worlds}(t)$ and $\mathsf{witn}(t) \subseteq \mathsf{witn}(s)$.

Definition 15 (top and bot)**.**

- $\mathsf{top} \coloneqq \langle W, \emptyset \rangle$
- $\mathsf{bot} \coloneqq \langle \emptyset, D^\star \rangle$

Fact 7. For any state s: $\mathsf{bot} \geq s \geq \mathsf{top}$

The extension relation will be used in the support definition, more specifically in the clause for implication: a state s supports an implication iff every extension of s that supports the antecedent, supports the consequent as well.

Before turning to the definition of support, however, we have to introduce one more auxiliary notion, namely that of a *witness feed*. The role of these witness feeds will be similar to that of assignments: they will be used to store certain information in evaluating whether or not a certain state supports a certain sentence. In particular, they play a role in evaluating existentially quantified sentences in the scope of one or more universal quantifiers. This will be further explained once we have specified the support relation.

Definition 16 (Witness feeds)**.**

1. A *witness feed* $\vec{e}$ is an n-tuple of objects in D, $n \geq 0$.
2. We denote the 0-tuple in D^0 by ϵ and call it *the empty witness feed*.
3. If $\vec{e} = \langle d_1, \ldots, d_n \rangle \in D^n$ is a witness feed, then, the witness feed $\vec{e}$ *augmented with* d, $\vec{e}^\frown d = \langle d_1, \ldots, d_n, d \rangle \in D^{n+1}$.

2.2 Support

We now have all the necessary ingredients to state the support relation.

Definition 17 (Support in INQW).
Let s be a $\mathbb{D}$-state, g an assignment, $\vec{e}$ a witness feed, and φ a formula in $\mathcal{L}$.

1. $s \models_{g,\tilde{e}} \varphi$ for atomic φ iff

   (i) $\forall M \in \mathsf{worlds}(s) : M \models_g \varphi$ and

   (ii) for every function symbol f occurring in φ: $I(f) \in \mathsf{witn}(s)$
2. $s \models_{g,\tilde{e}} \bot$ iff $\mathsf{worlds}(s) = \emptyset$
3. $s \models_{g,\tilde{e}} \varphi \wedge \psi$ iff $s \models_{g,\tilde{e}} \varphi$ and $s \models_{g,\tilde{e}} \psi$
4. $s \models_{g,\tilde{e}} \varphi \vee \psi$ iff $s \models_{g,\tilde{e}} \varphi$ or $s \models_{g,\tilde{e}} \psi$
5. $s \models_{g,\tilde{e}} \varphi \rightarrow \psi$ iff $\forall t \geq s$: if $t \models_{g,\tilde{e}} \varphi$ then $t \models_{g,\tilde{e}} \psi$
6. $s \models_{g,\tilde{e}} \forall x.\varphi$ iff $s \models_{g[x/d],\tilde{e}\frown d} \varphi$ for all $d \in D$
7. $s \models_{g,\tilde{e}} \exists x.\varphi$ iff $s \models_{g[x/\delta(\tilde{e})],\tilde{e}} \varphi$ for some $\delta \in \mathsf{witn}(s)$

The clauses that have changed w.r.t. INQB are those for atomic formulas, implication, universal quantification, and existential quantification. Let us look at these four clauses in some detail.

Atoms. For a state s to support an atomic sentence φ, the sentence has to be true in all worlds in $\mathsf{worlds}(s)$, as before, but moreover, for every function symbol f occurring in φ, $I(f)$ must be available as a witness in $\mathsf{witn}(s)$. To illustrate this, consider a language with one binary predicate symbol R and two 0-place function symbols (i.e. individual constants) a and b, such that $I(a) = d_1$ and $I(b) = d_2$. Then a state s supports the sentence Rab if and only if (i) for every $M \in \mathsf{worlds}(s)$ we have that $\langle d_1, d_2 \rangle \in M(R)$, and (ii) d_1 and d_2 are available as witnesses in $\mathsf{witn}(s)$.

Recall that in uttering a sentence, a speaker proposes to update the common ground of the conversation in such a way that it comes to support the sentence. Thus, in particular, in uttering Rab, a speaker proposes to add d_1 and d_2 to the witness set of the common ground. In this sense, we can think of atomic sentences like Rab as introducing new witnesses. We will see that other sentences, in particular existentials, may request a response that introduces new witnesses.

Implication. In order to determine whether a state s supports an implication $\varphi \rightarrow \psi$ we have to consider all extensions t of s that support φ. An extension t of s is a state such that $\mathsf{worlds}(t) \subseteq \mathsf{worlds}(s)$ and $\mathsf{witn}(t) \supseteq \mathsf{witn}(s)$. Thus, it may be that all the extensions of s that support φ contain certain witnesses that are not contained in s itself. This means that if ψ requires certain witnesses, as long as we need to introduce them to support φ, it is not necessary for s as such to already contain them for the implication to be supported in s.

To illustrate this, let us show that $\mathsf{top} \models_{g,\epsilon} Pa \rightarrow \exists x.Px$. Given the atomic clause, every $t \geq \mathsf{top}$ that supports Pa must be such that $I(a) \in \mathsf{witn}(t)$. In other words, every $t \geq \mathsf{top}$ that supports Pa contains a witness, namely $I(a)$, which is known to have the property P. It follows that $t \models_{g,\epsilon} \exists x.P(x)$, which in turn means that $\mathsf{top} \models_{g,\epsilon} Pa \rightarrow \exists x.Px$, even though top itself does not contain any witnesses.

Universal quantification. The clause for universal quantification is very much like the clause we had in INQB. Only now the witness feed plays a role as well. In determining whether a state s supports a formula $\forall x.\varphi$ we do not only set the current assignment g to $g[x/d]$, but we simultaneously augment the current witness feed $\vec{e}$ with the same object d. Then we check whether φ is supported by s relative to the adapted assignment and the augmented witness feed. As we will see below, the augmented witness feed is put to use in particular when φ contains an existential quantifier.

Existential quantification. At first sight, the clause for existential quantification is very much like the clause we had in INQB. But there is a crucial difference. Instead of checking whether $s \models_{g[x/d],\tilde{e}} \varphi$ holds for some object $d \in D$, we have to check whether $s \models_{g[x/\delta(\tilde{e})],\tilde{e}} \varphi$ holds for some *witness* $\delta \in \mathsf{witn}(s)$. Thus, as desired, support of an existentially quantified sentence $\exists x.Px$ now really requires the presence of a witness which is known to have the property P. This means that in uttering $\exists x.Px$, a speaker requests a response from other participants that introduces a suitable witness and then establishes of this witness that it has the property P.

Example 6 (Interaction between existentials and universals)**.** Consider the sentence $\forall x.\exists y.Rxy$. In order to determine whether $s \models_{g,\epsilon} \forall x.\exists y.Rxy$, we have to check whether $s \models_{g[x/d],\langle d\rangle} \exists y.Rxy$ for all $d \in D$. And this means that we have to verify whether for every $d \in D$, there is some $f \in \mathsf{witn}(s)$ such that $s \models_{g[x/d][y/f(d)],\langle d\rangle} Rxy$. This, then, is how universal and existential quantifiers interact: universal quantifiers add objects to the witness feed, and these objects then serve as the input for functional witnesses that may be needed for existentials in the scope of the universal. In this way, the witness that is required for the embedded existential in $\forall x.\exists y.Rxy$ may functionally depend on the value that the current assignment assigns to x.

Now let us take a step back, and make some general observations about the support relation. First of all, support is *persistent*. That is, if a state s supports a formula φ relative to a certain assignment g and a certain witness feed $\vec{e}$, then any extension of s also supports φ relative to g and $\vec{e}$.

Fact 8 (Persistence)**.** If $s \models_{g,\tilde{e}} \varphi$ and $t \geq s$, then $t \models_{g,\tilde{e}} \varphi$

As before, the semantics can be related to that of classical first-order logic: a model M classically satisfies a formula φ relative to an assignment g if and only if the state $\langle\{M\}, D^\star\rangle$ supports φ relative to g.[3]

Fact 9 (Singleton states with access to all witnesses behave classically)**.**
For all M, φ, g and $\vec{e}$:

$$\langle\{M\}, D^\star\rangle \models_{g,\tilde{e}} \varphi \quad\Leftrightarrow\quad M \models_{\,g} \varphi \quad \text{classically}$$

The semantics can also be related to that of INQB: a set of $\mathbb{D}$-models V supports a formula φ relative to an assignment g in INQB if and only if the state $\langle V, D^\star\rangle$ supports φ relative to g in INQW.[4]

Fact 10 (States with access to all witnesses behave as in INQB)**.**
For all φ, g and $\vec{e}$, and for every set of $\mathbb{D}$-models V:

$$\langle V, D^\star\rangle \models_{g,\tilde{e}} \varphi \quad\Leftrightarrow\quad V \models_g \varphi \quad \text{in INQB}$$

The maximal state, bot, supports all formulas, relative to all assignments and all witness feeds.

Fact 11 (bot supports all formulas)**.** For all φ, g, and $\vec{e}$: $\quad \mathsf{bot} \models \varphi$.

As before, we take $\neg\varphi$ to be an abbreviation of $\varphi \to \bot$, and $!\varphi$ an abbreviation of $\neg\neg\varphi$. Given fact 9, then, the derived clauses for $\neg\varphi$ and $!\varphi$ read as follows.

Fact 12 (Support for negation)**.**

- $s \models_{g,\tilde{e}} \neg\varphi$ iff for all $M \in \mathsf{worlds}(s)$: $M \not\models_g \varphi$ classically
- $s \models_{g,\tilde{e}} !\varphi$ iff for all $M \in \mathsf{worlds}(s)$: $M \models_g \varphi$ classically

2.3 Propositions, entailment, and equivalence

Based on the notion of support, we define the proposition expressed by a formula, and the notions of entailment and equivalence. Recall that our definitions assume a fixed first-order language $\mathcal{L}$ and a fixed proto-model $\mathbb{D} = \langle D, I\rangle$ for $\mathcal{L}$.

Definition 18 (Propositions)**.** The *proposition* expressed by φ relative to an assignment g is the set of all states that support φ relative to g and ϵ:

$$[\varphi]_g = \{s \in \mathcal{S}_{\mathbb{D}} \mid s \models_{g,\epsilon} \varphi\}$$

Definition 19 (Entailment and equivalence)**.**

- $\varphi \models \psi$ iff for all s and g: if $s \models_{g,\epsilon} \varphi$, then $s \models_{g,\epsilon} \psi$
- $\varphi \equiv \psi$ iff $\varphi \models \psi$ and $\psi \models \varphi$

[3]Witness feeds do not play a role in this case: since $\langle\{M\}, D^\star\rangle$ has access to all witnesses, we either have that $\langle\{M\}, D^\star\rangle \models_{g,\tilde{e}} \varphi$ for *all* witness feeds, or for *none*.

[4]Again, witness feeds do not play a role here, since $\langle\{M\}, D^\star\rangle$ has access to all witnesses.

Of course, entailment and equivalence can also be characterized in terms of propositions rather than directly in terms of support.

Fact 13 (Entailment and equivalence in terms of propositions)**.**

- $\varphi \models \psi$ iff for all g: $[\varphi]_g \subseteq [\psi]_g$
- $\varphi \equiv \psi$ iff for all g: $[\varphi]_g = [\psi]_g$

We also introduce notions of *factive* support, entailment, and equivalence, which ignore witness issues. These notions will be useful for several purposes.[5]

Definition 20 (Factive support, entailment, and equivalence)**.**

- A state s *factively supports* φ relative to $g, \vec{e}$ iff $s^\star \models_{g,\tilde{e}} \varphi$
- φ *factively entails* ψ if for all $s, g, \vec{e}$: if $s^\star \models_{g,\tilde{e}} \varphi$, then $s^\star \models_{g,\tilde{e}} \psi$
- φ and ψ are *factively equivalent* iff they factively entail each other.

2.4 Informativeness and inquisitiveness

As before, we define the informative content of a sentence φ relative to an assignment g as the set of worlds that are contained in at least one state that supports φ relative to g.

Definition 21 (Informative content)**.** $\mathsf{info}_g(\varphi) = \bigcup\{\mathsf{worlds}(s) \mid s \in [\varphi]_g\}$

Also as before, the informative content of a sentence φ relative to an assignment g always coincides with the *truth set* of φ relative to g, $|\varphi|_g$, i.e., the set of worlds that satisfy φ in classical first-order logic relative to g. So as far as informative content is concerned, INQW does not diverge from classical first-order logic.

Fact 14 (INQW preserves classical treatment of informative content)**.**
For every φ and every g: $\mathsf{info}_g(\varphi) = |\varphi|_g$

In terms of the informative content of a formula, we define whether it is informative and/or inquisitive. These definition are parallel to the ones we had in INQB, only now we add a distinction between *factual* inquisitiveness and *witness* inquisitiveness.

Definition 22 (Inquisitiveness and informativeness in a state)**.**

- φ is *informative* in s w.r.t. g iff $\mathsf{worlds}(s) \cap \mathsf{info}_g(\varphi) \neq \mathsf{worlds}(s)$
- φ is *inquisitive* in s w.r.t. g iff $\langle \mathsf{worlds}(s) \cap \mathsf{info}_g(\varphi), \mathsf{witn}(s)\rangle \not\models_{g,\epsilon} \varphi$
- φ is *factively inquisitive* in s w.r.t. g iff it is inquisitive in $s^\star$ w.r.t. g
- φ is *witness inquisitive* in s w.r.t. g iff it is inquisitive in $s^\emptyset$ w.r.t. g

[5] When considering the definition of these notions, recall that $s^\star$ is defined as the state that is just like s, except that it contains $D^\star$ as its witness set.

As before, these relative notions also given rise to the following absolute notions.

Definition 23 (Absolute inquisitiveness and informativeness)**.**

- φ is *informative* w.r.t. g iff it is informative in some s w.r.t. g
- φ is *inquisitive* w.r.t. g iff it is inquisitive in some s w.r.t. g
- φ is *factively inquisitive* w.r.t. g iff for some s, φ is inquisitive in $s^\star$ w.r.t. g
- φ is *witness inquisitive* w.r.t. g iff for some s, φ is inquisitive in $s^\emptyset$ w.r.t. g

Note that because of persistence, rather than stating these absolute notions in terms of "some s" we could also state them in terms of top. In terms of these notions of informativeness and inquisitiveness, we may distinguish several semantic categories. In defining these categories we will suppress reference to assignments. So, strictly speaking, the definitions only apply to sentences without free variables. Of course, they can easily be generalized so as to apply to formulas in general.

Definition 24 (Questions, assertions, and hybrids)**.**

- φ is a *question* iff it is not informative
- φ is an *assertion* iff it is not inquisitive
- φ is a *hybrid* iff it is both inquisitive and informative

Definition 25 (Tautologies and contradictions)**.**

- φ is a *tautology* iff it is neither informative nor inquisitive
- φ is a *factive tautology* iff $\mathsf{info}(\varphi) = W$
- φ is a *contradiction* iff $\mathsf{info}(\varphi) = \emptyset$

Definition 26 (Non-trivial questions and assertions)**.**

- φ is a *non-trivial question* iff it is inquisitive and not informative
- φ is a *non-trivial assertion* iff it is informative and not inquisitive

Definition 27 (Witness questions and factive questions)**.**

- φ is a *witness question* iff it is not informative, and witness inquisitive
- φ is a *factive question* iff it is not informative, and factively inquisitive

Definition 28 (Factive hybrids and assertions)**.**

- φ is a *factive hybrid* iff it is informative and factively inquisitive
- φ is a *factive assertion* iff it is informative and not factively inquisitive

Given these semantic categories, we can prove a number of facts which give further insight into the system.

Fact 15 (The propositional case)**.**
Let φ be a formula containing only 0-place predicate symbols. Then:

- φ is a non-trivial question iff it is a factive question
- φ is a non-trivial assertion iff it is a factive assertion
- φ is a hybrid iff it is a factive hybrid
- φ is a tautology iff it is a factive tautology

Fact 16 (Atomic sentences)**.** Every atomic sentence is a factive assertion.

Notice that atomic sentences containing one or more function symbols are always *witness* inquisitive. However, they are never *factively* inquisitive.

Fact 17 (Some sufficient syntactic conditions for (factive) assertionhood)**.**

1. If φ is an atomic sentence, then:
   - φ is a factive assertion, and
   - φ is an assertion iff there are no occurrence of function symbols in φ
2. $\bot$ is an assertion;
3. If φ, ψ are (factive) assertions, then so is $\varphi \wedge \psi$;
4. If ψ is a (factive) assertion, then so is $\varphi \rightarrow \psi$;
5. If φ is a (factive) assertion, then so is $\forall x.\varphi$.

Fact 18. For any φ: $!\varphi$ is an assertion and $?\varphi$ is a question.

Fact 19 (Inquisitiveness and existential quantification)**.**

- For any φ, $\exists x.\varphi$ is witness inquisitive.
- An extreme case: $\exists x.x = x$ is a factive tautology, but not a tautology *tout court*, because it is witness inquisitive.

2.5 The boundedness problem

Now that we have investigated the logical properties of INQW in some detail, let us return to the main problem that we set out to resolve. The problem was that Ciardelli's boundedness formulas were semantically indistinguishable in INQB. They were supported by exactly the same states. As a result, it was impossible in INQB to capture the intuition that these formulas license a different range of responses.

This problem no longer arises in INQW. In particular, the boundedness formulas are no longer semantically equivalent.

Fact 20 (The boundedness formulas)**.** The boundedness formula and the positive boundedness sentence are not equivalent in INQW.

Proof. Consider a state s such that:

- $\mathsf{worlds}(s) = \{w\}$, where $w(P) = \{0\}$
- $\mathsf{witn}(s) = \{0\}$

This state factively supports both $\exists x.B(x)$ and $\exists x.(x > 0 \wedge B(x))$. However, while the boundedness formula is supported in s *tout court*, $s \models \exists x.B(x)$, the positive boundedness formula is not, $s \not\models \exists x.(x > 0 \wedge B(x))$. So, the boundendness sentence and the positive boundedness sentence are not equivalent in INQW (although they *are* factually equivalent). □

2.6 Licensing

In order to fully address Ciardelli's challenge, we do not only have to show that the boundedness formulas are semantically distinguishable in INQW; we also need to specify a sensible formal notion of *licensing*, under which the boundedness formulas indeed license different responses. In particular, the response $B(0)$ should be licensed by the boundedness formula itself, but not by the positive boundedness formula.

The main intuition about licensing that can be found in previous work on inquisitive semantics (e.g, Groenendijk and Roelofsen, 2009), and also in earlier work on the semantics and pragmatics of questions and answers (e.g., Groenendijk and Stokhof, 1984; Groenendijk, 1999), is that an initiative φ licenses a response ψ just in case: (i) ψ provides enough information to resolve the issue raised by φ, and (ii) ψ does not provide *more* information than is needed to resolve the issue raised by φ. To illustrate this idea, consider the question in (1) and the responses in (1-a-c).

(1) Is Mary going to the party?
   a. Yes, she is going.
   b. Cats don't like brocolli.
   c. Yes, she is going, and cats don't like brocolli.

The intuition is that (1) licenses the response in (1-a), but not those in (1-b) and (1-c). The response in (1-b) is not licensed because it does not resolve the issue raised by (1), and the response in (1-c) is not licensed because it provides *more* information than is needed to resolve the issue raised by (1). Only (1-a) provides exactly enough information to resolve the given issue.

This intuitive notion of licensing can be made precise in INQW as follows.[6]

Definition 29 (Licensing)**.**
Let φ be inquisitive and let ψ be an assertion. Then:

1. φ is a *issue-resolving response* to ψ iff $\varphi \models \psi$.

[6]For simplicity, we restrict our attention here to the case where the initiative is inquisitive and the response is an assertion. Presumably, the notion can be generalized in a natural way so as to apply to arbitrary initiatives and responses.

2. φ licenses ψ iff ψ is an issue-resolving response to φ, and there is no other issue-resolving response χ to φ such that $\psi \models \chi$.

According to this notion of licensing, $B(0)$ is indeed licensed by the boundedness formula itself, but not by the positive boundedness formula.

Fact 21 (Responses licensed by the boundedness formulas)**.**

- $\exists x.Bx$ licenses $B(n)$ for any $n \geq 0$
- $\exists x.(x \neq 0 \wedge Bx)$ licenses $B(n)$ for any $n > 0$, but not for $n = 0$

Finally, we note that one licensed response may intuitively be preferred over another. For instance, $B(1)$ and $B(135)$ are both licensed responses to $\exists x.Bx$. However, $B(1)$ is intuitively preferred over $B(135)$: if the information state of the responder supports $B(1)$ then it would be misleading for her to actually choose $B(135)$ as a response. In general, if ψ and χ are two licensed responses to φ, and ψ factually entails χ, then ψ is preferred over χ as a response to φ.

Definition 30 (Comparing licensed responses)**.**
Let φ be an inquisitive initiative, let ψ and χ be two licensed responses to φ, and let σ be an information state, i.e., a set of worlds. Then:

1. ψ is *preferred* over χ as a response to φ iff ψ factively entails χ.
2. ψ is an *optimal response* to φ in σ iff
   - ψ is a licensed response to φ,
   - $\sigma \subseteq \mathsf{info}(\psi)$, and
   - for every licensed response ξ to φ that is preferred over ψ, $\sigma \not\subseteq \mathsf{info}(\xi)$.

To illustrate the notion of an optimal response, consider an information state consisting of three worlds, one where the highest element of P is 5, one where it is 14, and one where it is 3. The optimal response to $\exists x.Bx$ in this information state is $B(14)$. This accounts for the intuition that, on the one hand, any response $B(n)$ with $n < 14$, even though licensed, would be *qualitatively* inappropriate, while any response $B(n)$ with $n > 14$ would be *quantitively* dispreferred. The only optimal response in this scenario is $B(14)$.

3 Conclusion

We have seen in this paper that it is possible to construe a support-based first-order inquisitive semantics which avoids the boundedness problem described by Ciardelli (2009, 2010). The central idea was that a state supports an existentially quantified sentence $\exists x.Px$ just in case there is a specific witness in that state which is known to have the property P. Once this idea is explicitly incorporated into the system, all the familiar notions like that of informative and inquisitive sentences, entailment, equivalence, etc. can be defined in the usual way. The only additional distinction that we get is that between *factive* inquisitiveness and *witness* inquisitiveness. Most importantly, the resulting system allows us to define a natural notion of licensing, which makes exactly the desired predictions for Ciardelli's boundedness formulas.

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