First-Order Inquisitive Pair Logic

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Abstract. We introduce two different calculi for a first-order extension of inquisitive pair semantics (Groenendijk 2008): Hilbert-style calculus and Tree-sequent calculus. These are first-order generalizations of (Mascarenhas 2009) and (Sano 2009), respectively. First, we show the strong completeness of our Hilbert-style calculus via canonical models. Second, we establish the completeness and soundness of our Tree-sequent calculus. As a corollary of the results, we semantically establish that our Tree-sequent calculus enjoys a cut-elimination theorem.

1 Introduction

Groenendijk [1] first introduced the *inquisitive pair semantics* for a language of propositional logic to capture both classical and inquisitive meanings of a sentence. For example, the classical meaning of $\mathbf{p} \vee \mathbf{q}$ is $|\mathbf{p} \vee \mathbf{q}|$ and the inquisitive meaning of it is $\{\, |\mathbf{p}|, |\mathbf{q}| \,\}$, where $|A|$ is the set of all truth functions making A true. In the first logical study for inquisitive pair semantics [2], Mascarenhas revealed that the corresponding *inquisitive pair logic* is an axiomatic extension of intuitionistic logic (however, it is not closed under uniform substitutions) and established the completeness of it. Independently, following the idea of [3], the author gave a complete and cut-free Gentzen-style sequent calculus for inquisitive pair logic [4]. After these studies, Ciardelli and Roelofsen [5] generalized inquisitive pair semantics within the propositional level and revealed that their *generalized inquisitive logic* has various beautiful logical properties.

Disjunction $\vee$ allows us to formalize an English sentence containing 'or'. However, in order to handle the sentences containing quantifications as well as 'which', 'who', etc., we need a first-order extension of inquisitive semantics. Ciardelli [6] studied how to give a recursive definition of inquisitive meaning in a first-order setting. As far as the author knows, however, there is no complete axiomatization of first-order inquisitive logic, though there was a preliminary study toward this direction [7, Ch.6]. This paper contributes to this point. In this paper, we focus on a first-order extension of the original *inquisitive pair semantics* and give two different complete calculi for a *first-order inquisitive pair logic*: Hilbert-style calculus and Gentzen-style sequent calculus. We can regard these as first-order generalizations of [2] and [4], respectively.

There are various ways of considering first-order extensions of intuitionistic logic for Kripke semantics: e.g. by expanding the domain or keeping it constant. Following [7, Ch.6], this paper also concerns the constant-domain semantics, which means that we adopt **CD**: $\forall x.(A \vee B(x)) \to (A \vee \forall x. B(x))$ (x is not free in A) as our logical axiom. In the first part of this paper, we establish the correspondence between the first-order inquisitive models and a specific class of constant-domain Kripke models (Theorem 1). After introducing the Hilbert-style axiomatization of first-order inquisitive pair logic, we use the correspondence above and the canonical model method [8, Ch.7.2] to establish the strong completeness (Corollary 1). In the second part, we first extend the sequent calculus of [4] to cover the quantifiers (**CD** gives us the simpler rule, cf. [3,9]), and then, we establish the completeness (Theorem 3) and soundness (Theorem 5) of our Tree-sequent calculus. By combining these with the results of the first part, we can semantically establish the cut-elimination theorem of our sequent calculus.

In the propositional level, the generalized inquisitive logic is a 'limit' of a hierarchy of inquisitive logics [7, Ch.6], one of which is the inquisitive pair logic. Therefore, based on this study, the author hopes that we could also 'approximate' a generalized first-order inquisitive logic by considering the corresponding first-order hierarchy.

2 Inquisitive Semantics and Constant-Domain Kripke Semantics

2.1 Inquisitive Pair Semantics

Our syntax $\mathcal{L}$ consists of a countable set $\mathsf{VAR} = \{ x_i \mid i \in \omega \}$ of variables, a countable set $\{ c_i \mid i \in \omega \}$ of constant symbols, a countable set of predicate symbols P, the propositional connectives: $\bot$, $\neg$, $\to$, $\wedge$, $\vee$, the quantifiers: $\forall$, $\exists$, and the parentheses: (,). t is a *term* if t is a variable or a constant symbol. Then, the *formulas* of $\mathcal{L}$ are defined as usual. We use Γ and Δ, etc. to denote a (possibly infinite) set of formulas. For a finite Γ, $\bigwedge \Gamma$ (or, $\bigvee \Gamma$) is defined as the conjunction (or, disjunction) of all formulas of Γ, if Γ is non-empty; otherwise $\top$ (or, $\bot$, respectively). $A[t/x]$ denotes the result of the simultaneous substitution of t for all free occurrences of x in A.

An (*first-order*) *inquisitive model* $\mathfrak{M}$ consists of a non-empty set W, a non-empty set D, and a valuation V satisfying $c^V \in D$ and $P^V_w \subseteq D^n$ ($w \in W$), where n is the arity of P[1]. Given any $W \neq \emptyset$, we say that $s \subseteq W$ is *pairwise* if $\#s \leq 2$ and $s \neq \emptyset$. Given any inquisitive model $\mathfrak{M} = \langle W, R, D \rangle$, any pairwise $s \subseteq W$, any *assignment* $g : \mathsf{VAR} \to D$, and any formula A, the satisfaction relation $s, g \models_{\mathfrak{M}} A$ is defined by:

$$
\begin{array}{lll}
s, g \models_{\mathfrak{M}} P(t_1, \dots, t_n) & \text{iff} & \langle \overline{g}(t_1), \dots, \overline{g}(t_n) \rangle \in P^V_w \text{ for any } w \in s; \\
s, g \models_{\mathfrak{M}} \bot & & \text{Never ;} \\
s, g \models_{\mathfrak{M}} \neg A & \text{iff} & \text{for any pairwise } s' \subseteq s\text{: } s', g \not\models_{\mathfrak{M}} A; \\
s, g \models_{\mathfrak{M}} A \wedge B & \text{iff} & s, g \models_{\mathfrak{M}} A \text{ and } s, g \models_{\mathfrak{M}} B; \\
s, g \models_{\mathfrak{M}} A \vee B & \text{iff} & s, g \models_{\mathfrak{M}} A \text{ or } s, g \models_{\mathfrak{M}} B; \\
s, g \models_{\mathfrak{M}} A \to B & \text{iff} & \text{for any pairwise } s' \subseteq s\text{: } s', g \models_{\mathfrak{M}} A \text{ implies } s', g \models_{\mathfrak{M}} B; \\
s, g \models_{\mathfrak{M}} \forall x. A & \text{iff} & \text{for any } a \in D\text{: } s, g(x|a) \models_{\mathfrak{M}} A; \\
s, g \models_{\mathfrak{M}} \exists x. A & \text{iff} & \text{for some } a \in D\text{: } s, g(x|a) \models_{\mathfrak{M}} A,
\end{array}
$$

where $\overline{g}(t) := g(x)$ (if $t \equiv x$); c^V (if $t \equiv c$), and $g(x|a)$ is the x-*variant* of g such that $g(x|a)(x) = a$. We usually drop the subscript $\mathfrak{M}$ from $\models_{\mathfrak{M}}$, if it is clear from the context.

[1] For a propositional variable $\mathbf{p}$ (i.e. 0-ary predicate symbol), we define $\mathbf{p}^V_w \in \{ \mathbf{true}, \mathbf{false} \}$.

Given any $\mathfrak{M} = \langle W, D, V \rangle$, A is *valid in* $\mathfrak{M}$ (notation: $\models_{\mathfrak{M}} A$) if for any pairwise $s \subseteq W$ and for any $g : \mathsf{VAR} \to D$, $s, g \models_{\mathfrak{M}} A$. Let M be a class of inquisitive models. $\Gamma \models_{\mathsf{M}} A$ means that, for any $\mathfrak{M} \in \mathsf{M}$, any assignment g and any pairwise s, if $s, g \models_{\mathfrak{M}} B$ for all $B \in \Gamma$ then $s, g \models_{\mathfrak{M}} A$. We say that A is *valid in* M (notation: $\mathsf{M} \Vdash A$) if $\emptyset \models_{\mathsf{M}} A$. Define M_{all} as the class of *all* inquisitive models.

In [6] and [7, Ch.6], the following special class of inquisitive models are considered: Let us fix $D \neq \emptyset$ and fix a mapping $\mathscr{I} \colon \{ c_i \mid i \in \omega \} \to D$, i.e., an *interpretation* on D of all the constant symbols. Let $W_{(D,\mathscr{I})}$ be the collection of all first-order classical structures for $\mathcal{L}$ such that the universe of $\mathfrak{A}$ is D and, $c^{\mathfrak{A}} = \mathscr{I}(c)$ for any $\mathfrak{A} \in W_{(D,\mathscr{I})}$. Define the valuation V of inquisitive model by: $c^V := c^{\mathfrak{A}}$ for some fixed $\mathfrak{A}$ and $P^V_{\mathfrak{A}} = P^{\mathfrak{A}}$. Then, $\langle W_{(D,\mathscr{I})}, D, V \rangle$ is an inquisitive model. Let us define that an *intended inquisitive model* is such a tuple $\langle W_{(D,\mathscr{I})}, D, V \rangle$ for some D and $\mathscr{I}$. Fix an assignment g. Remark that we can rewrite the satisfaction clause for atoms as follows: $s, g \models P(t_1, \ldots, t_n)$ iff $\mathfrak{A} \models P(t_1, \ldots, t_n)[g]$ for any $\mathfrak{A} \in s$, where $\mathfrak{A} \models A[g]$ means the ordinary *classical* satisfaction relation.

Definition 1. $\mathsf{M}_{\text{int}} = \{ \langle W_{(D,\mathscr{I})}, D, V \rangle \mid D \neq \emptyset \textit{ and } \mathscr{I} : \{ c_i \mid i \in \omega \} \to D \}$.

So, M_{int} is the class of all intended inquisitive models. We will show that there is no difference between M_{all} and M_{int} with respect to the logical consequence (Theorem 1).

Let us explain why this paper studies first-order inquisitive pair semantics: While inquisitive pair semantics shows a peculiar logical-phenomena in calculating the inquisitive meaning of $\mathbf{p} \vee \mathbf{q} \vee \mathbf{r}$ (i.e. all the *possibilities* (defined below) for $\mathbf{p} \vee \mathbf{q} \vee \mathbf{r}$) in the propositional level, it still forms a good starting point to investigate *first-order inquisitive logic*, i.e, all valid formulas on M_{int} in first-order inquisitive semantics [7, Ch.6] by Ciardelli. In what follows in this subsection, let us pay attention only to M_{int}. Before explaining the detail above, we would like to introduce some terminology. Define that $s \subseteq W_{(D,\mathscr{I})}$ is *n-tuplewise* if $1 \leq \#s \leq n$. '2-tuplewise' is the same notion as 'pairwise'. If we replace 'pairwise' with 'n-tuplewise' or 'non-empty' in the satisfaction clauses above, then we obtain *first-order inquisitive n-tuple semantics* or *first-order inquisitive semantics* [7, Ch.6] by Ciardelli[2], respectively.

Consider the propositional counterpart of our inquisitive pair semantics and define that a *possibility* for a propositional formula A is a $\supseteq$-maximal element s such that $s \models A$ (cf. [1]). Denote all the possibilities for A by $[A]$. Then, $[\mathbf{p} \vee \mathbf{q}] = \{ |\mathbf{p}|, |\mathbf{q}| \}$ holds, where $|A|$ is all the truth functions making A true. Ciardelli, however, showed that $[\mathbf{p} \vee \mathbf{q} \vee \mathbf{r}] \neq \{ |\mathbf{p}|, |\mathbf{q}|, |\mathbf{r}| \}$ in inquisitive pair semantics [7, Ch.5]). Inquisitive 3-tuplewise semantics can fix this defeat for $\mathbf{p} \vee \mathbf{q} \vee \mathbf{r}$. However, in order to avoid such peculiar phenomena for any formula containing $\vee$, we should drop the cardinality restriction of the upper bound of $\#s$ in the satisfaction clauses above. Such a consideration leads us to (propositional) inquisitive semantics by Ciardelli and Roelofsen [5].

Let InqQL_n (or, InqQL) be all the valid formulas on M_{int} in first-order inquisitive n-tuplewise semantics (or, first-order inquisitive semantics, respectively). Let InqL_n and InqL be their propositional counterparts. Then, $\bigcap_{2 \leq n} \mathsf{InqL}_n = \mathsf{InqL}$ holds [7, Corollary

[2] Ciardelli also observed that the restriction $\#s \leq 2$ gives us the equivalent semantics to the original inquisitive pair semantics by Groenendijk (see [7, Ch.5, pp.55-6]). In this sense, we still call our semantics '(first-order) inquisitive *pair* semantics'.

4.1.6.], and so, InqL_2 forms a starting point of approximating InqL. When we move to the first-order level, we do not know whether $\bigcap_{2\leq n} \mathsf{InqQL}_n = \mathsf{InqQL}$ in this stage. However, it is obvious that $\bigcap_{2\leq n} \mathsf{InqQL}_n \subseteq \mathsf{InqQL}$. Therefore, first-order inquisitive pair semantics still forms a good starting point to investigate InqQL.

2.2 Constant-Domain Kripke Semantics

If we extend the first-order intuitionistic logic **IQL** with the axiom **CD** in Table 1 below, then we can obtain the following simpler Kripke semantics [8, Ch.3.4]. A *constant-domain Kripke model* (in short: *cd-model*) is a tuple $\langle W, \leq, D, V \rangle$, where $W \neq \emptyset$, $\leq$ on W is a pre-order, $D \neq \emptyset$, and V is a valuation satisfying $c^V \in D$, $P^V_w \subseteq D^n$, and $P^V_w \subseteq P^V_v$ if $w \leq v$ (the *hereditary condition*). Given any cd-model $\langle W, \leq, D, V \rangle$, any $g : \mathsf{VAR} \to D$, $w \in W$, and any A of $\mathcal{L}$, the satisfaction relation $\Vdash$ is defined by:

$$
\begin{array}{lll}
\mathfrak{M}, w, g \Vdash P(t_1, \ldots, t_n) & \text{iff} & \langle \overline{g}(t_1), \ldots, \overline{g}(t_n) \rangle \in P^V_w; \\
\mathfrak{M}, w, g \Vdash \bot & & \text{Never ;} \\
\mathfrak{M}, w, g \Vdash \neg A & \text{iff} & \text{for any } w' \geq w\text{: } \mathfrak{M}, w', g \nVdash A; \\
\mathfrak{M}, w, g \Vdash A \wedge B & \text{iff} & \mathfrak{M}, w, g \Vdash A \text{ and } \mathfrak{M}, w, g \Vdash B; \\
\mathfrak{M}, w, g \Vdash A \vee B & \text{iff} & \mathfrak{M}, w, g \Vdash A \text{ or } \mathfrak{M}, w, g \Vdash B; \\
\mathfrak{M}, w, g \Vdash A \to B & \text{iff} & \text{for any } w' \geq w\text{: } w', g \Vdash A \text{ implies } w', g \Vdash B; \\
\mathfrak{M}, w, g \Vdash \forall x. A & \text{iff} & \text{for any } a \in D\text{: } \mathfrak{M}, w, g(x|a) \Vdash A; \\
\mathfrak{M}, w, g \Vdash \exists x. A & \text{iff} & \text{for some } a \in D\text{: } \mathfrak{M}, w, g(x|a) \Vdash A.
\end{array}
$$

Given any cd-model $\mathfrak{M} = \langle W, \leq, D, V \rangle$, A is *valid in* $\mathfrak{M}$ (notation: $\mathfrak{M} \Vdash A$) if for any $w \in W$ and for any $g : \mathsf{VAR} \to D$, $\mathfrak{M}, w, g \Vdash A$. By the following procedure, we can

Table 1. All Additional Axioms for First-Order Inquisitive Pair Logic

CD	$\forall x. (A \vee B(x)) \to (A \vee \forall x. B(x))$, where x is not free in A.
H2	$A \vee (A \to B \vee \neg B)$
W2	$(A \to B) \vee (B \to A) \vee ((A \to \neg B) \wedge (B \to \neg A))$
ADN	$\neg\neg P(t_1, \ldots, t_n) \to P(t_1, \ldots, t_n)$ for any atomic $P(t_1, \ldots, t_n)$

regard any inquisitive model $\mathfrak{M} = \langle W, D, V \rangle$ as a cd-model $\langle W', \leq, D', V' \rangle$ for first-order intuitionistic logic with the axiom **CD**. Put $W' := \{ s \subseteq W \mid s \text{ is pairwise} \}$. Define a pre-order $\leq$ on W' by $s \leq t$ iff $t \subseteq s$. Define $D' := D$. As for the valuation V', we define $c^{V'} = c^V$ and $\langle d_1, \ldots, d_n \rangle \in P^{V'}_s$ iff $\langle d_1, \ldots, d_n \rangle \in P^V_w$ for any $w \in s$ (s: pairwise). It is easy to see that V satisfies the hereditary condition. Then, we can show that $s, g \models_{\mathfrak{M}} A$ iff $\langle W', \leq, D', V' \rangle, s, g \Vdash A$, for any pairwise $s \subseteq W$ and any A. This observation allows us to say that all theorems of first-order intuitionistic logic as well as **CD** are valid in any inquisitive model.

Moreover, we can specify the class of cd-models corresponding to $\mathsf{M}_{\mathrm{all}}$ as Mascarenhas [2] did for the propositional language. $\langle W', \leq, D' \rangle$ satisfies:

($h2$**)** the maximum length of $\leq$-chains is 2 (or, it is of depth ≤ 2, simply);
($w2$**)** each state can have no more than two distinct successors.

These observations tells us that both **H2** and **W2** in Table 1 are valid on any inquisitive model $\langle W, D, V \rangle$ by ($h2$) and ($w2$), respectively (see [2, Theorem 35]). There is one more feature of the above $\langle W', \leq, D', V' \rangle$:

Definition 2. $\mathfrak{M} = \langle W, \leq, D, V \rangle$ *has the* intersection property *if, for any* $w \in W$, $P_w^V = \bigcap \left\{ P_v^V \mid w \leq v \text{ and } v \text{ is an endpoint} \right\}$.

This feature validates the axiom **ADN** in Table 1 on any inquisitive model:

Proposition 1. *Let* $\mathfrak{M} = \langle W, \leq, D, V \rangle$ *be a Kripke model such that* $\{ v \mid w \leq v \}$ *is finite* $(w \in W)$ *and* $\mathfrak{M}$ *satisfies the intersection property. Then,* **ADN** *is valid in* $\mathfrak{M}$.

Proof. Fix any $w \in W$ and any assignment g. Assume $\mathfrak{M}, w, g \Vdash \neg\neg P(t_1, \ldots, t_n)$. We show $\mathfrak{M}, w, g \Vdash P(t_1, \ldots, t_n)$. By assumption, for any $v \geq w$, we can find some $u \geq v$ such that $\mathfrak{M}, u, g \Vdash P(t_1, \ldots, t_n)$. Since $\{ w' \mid w \leq w' \}$ is finite, we can find $u^* \geq w$ such that u^* is an endpoint. Then, $\mathfrak{M}, u^*, g \Vdash P(t_1, \ldots, t_n)$. By the intersection property, we can conclude that $\mathfrak{M}, w, g \Vdash P(t_1, \ldots, t_n)$, as desired. $\square$

Clearly, the above $\langle W', \leq, D', V' \rangle$ has the intersection property. Under $(h2)$ and $(w2)$, $\{ v \mid w \leq v \}$ is always finite $(w \in W)$. Therefore, **ADN** is valid in $\mathsf{M}_{\mathrm{all}}$.

Definition 3. *Let* VI *be the class of all cd-models satisfying* $(w2)$, $(h2)$ *and the intersection property.*

$\Gamma \Vdash_{\mathsf{VI}} A$ means that for any $\mathfrak{M} \in \mathsf{VI}$, any assignment g and any state w in $\mathfrak{M}$, if $\mathfrak{M}, w, g \models B$ for all $B \in \Gamma$ then $\mathfrak{M}, w, g \models A$. We denote $\emptyset \Vdash_{\mathsf{VI}} A$ by $\mathsf{VI} \Vdash A$. The following is a generalization of [2, Theorem 36] to this setting.

Theorem 1. $\Gamma \models_{\mathsf{M}_{\mathrm{all}}} A$ *iff* $\Gamma \models_{\mathsf{M}_{\mathrm{int}}} A$ *iff* $\Gamma \Vdash_{\mathsf{VI}} A$.

Proof. $\Gamma \Vdash_{\mathsf{VI}} A \Longrightarrow \Gamma \models_{\mathsf{M}_{\mathrm{all}}} A$ is clear from the above argument. By definition, $\Gamma \models_{\mathsf{M}_{\mathrm{all}}} A \Longrightarrow \Gamma \models_{\mathsf{M}_{\mathrm{int}}} A$. So, it suffices to show $\Gamma \models_{\mathsf{M}_{\mathrm{int}}} A \Longrightarrow \Gamma \Vdash_{\mathsf{VI}} A$. We establish the contrapositive implication. Assume $\Gamma \nVdash_{\mathsf{VI}} A$, i.e., there exists some cd-model $\mathfrak{M} \in \mathsf{VI}$, some w in $\mathfrak{M}$ and some g such that $\mathfrak{M}, w, g \Vdash B$ $(B \in \Gamma)$ and $\mathfrak{M}, w, g \nVdash A$. Take the point-generated submodel $\mathfrak{M}_w$ by w of $\mathfrak{M}$. It is easy to see that $\mathfrak{M}, w, g \Vdash C$ iff $\mathfrak{M}_w, w, g \Vdash C$ for any formula C. Thus, $\mathfrak{M}_w, w, g \Vdash B$ $(B \in \Gamma)$ and $\mathfrak{M}_w, w, g \nVdash A$. Since $(w2)$, $(h2)$ (and the intersection property) still hold in $\mathfrak{M}_w$, we can state that $\mathfrak{M}_w$ has one of the following shapes: (i) one point reflexive model; (ii) 'I'-shape; (iii) 'V'-shape. Write $\mathfrak{M}_w := \langle W, \leq, D, V \rangle$. First, consider the case (i). Define an interpretation $\mathscr{I}$ on D of constants by $\mathscr{I}(c) = c^V$. Define a first-order classical structure $\mathfrak{A}$ by: $|\mathfrak{A}| = D$, $c^{\mathfrak{A}} = \mathscr{I}(c)$, and $P^{\mathfrak{A}} = P_w^V$. Then, we can establish that $\mathfrak{M}_w, w, g \Vdash C$ iff $\{ \mathfrak{A} \}, g \models C$ for any formula C. Therefore, we have found $\mathfrak{A} \in W_{(D,\mathscr{I})}$ such that $\{ \mathfrak{A} \}, g \models B$ $(B \in \Gamma)$ and $\{ \mathfrak{A} \}, g \not\models A$, i.e., $\Gamma \not\models_{\mathsf{M}_{\mathrm{int}}} A$, as required. Second, consider the case (ii). We can put $W = \{ w, v \}$. By the intersection property, however, P_v^V are the same as P_w^V. So, we can reduce this case to the case (i). Third, let us consider (iii). Put $W = \{ w, v, u \}$. We regard v and u as the 'leaves' of the 'V'-shape tree with the root w. Similarly to (i), define an interpretation $\mathscr{I}$ on D of constants by $\mathscr{I}(c) = c^V$. In this case, however, we need to define two first-order classical structures $\mathfrak{A}$ and $\mathfrak{B}$ by: $|\mathfrak{A}| = |\mathfrak{B}| = D$, $c^{\mathfrak{A}} = c^{\mathfrak{B}} = \mathscr{I}(c)$, and $P^{\mathfrak{A}} = P_v^V$ and $P^{\mathfrak{B}} = P_u^V$. By induction, we can show that $\mathfrak{M}_w, w, g \Vdash C$ iff $\{ \mathfrak{A}, \mathfrak{B} \}, g \models C$ for any C. By the similar argument to (i), we can conclude that $\Gamma \not\models_{\mathsf{M}_{\mathrm{int}}} A$. $\square$

By this correspondence, we can easily show the following propositions (cf. [4]).

Proposition 2. *Let $s \subseteq W$ be pairwise and $w, v \in W$ distinct.* (i) *If $s, g \models A$ and $s' \subseteq s$ is pairwise, then $s', g \models A$;* (ii) $\{ w, v \}, g \models \neg A$ iff $\{ w \}, g \not\models A$ *and* $\{ v \}, g \not\models A$; (iii) $\{ w \}, g \models \neg A$ iff $\{ w \}, g \not\models A$; (iv) $\{ w \}, g \models A \to B$ iff $\{ w \}, g \models A$ *implies* $\{ w \}, g \models B$.

Let $\mathsf{M}_2 := \{ \langle W, D, V \rangle \mid \#W = 2 \}$, $\mathsf{M}_1 := \{ \langle W, D, V \rangle \mid \#W = 1 \}$ and $\mathsf{M}_{\geq 2} := \{ \langle W, D, V \rangle \mid \#W \geq 2 \}$.

Proposition 3. (i) *Assume that $\#W \geq 2$. Then, A is valid in an inquisitive model $\mathfrak{M}$* iff *$s, g \models A$ for any pairwise s with $\#s = 2$ and any g in $\mathfrak{M}$.* (ii) $\mathsf{M}_1 \models A$ iff *A is classically valid.* (iii) *If $\mathsf{M}_{\geq 2} \models A$, then A is classically valid.* (iv) $\mathsf{M}_{\mathrm{all}} \models A$ iff $s, g \models_{\langle W, D, V \rangle} A$ *for any pairwise $s \subseteq W$ with $\#s = 2$, any g, and any $\langle W, D, V \rangle \in \mathsf{M}_{\geq 2}$.*

3 A Complete Hilbert-style Calculus for Inquisitive Pair Logic

Definition 4. *Define* $\mathbf{QLV}^+$ *is* **IQL** *extended with all the axioms in Table 1.*

The reader can find the axiomatization of the first-order intuitionistic logic **IQL** in [10]. Define $\Gamma \vdash A$ if $\vdash \bigwedge \Gamma' \to A$ for some finite $\Gamma' \subseteq \Gamma$. If $\Gamma = \emptyset$, we write $\mathbf{QLV}^+ \vdash A$ but we usually drop '$\mathbf{QLV}^+$' from it and write $\vdash A$, when no confusion arises. In order to show the completeness of $\mathbf{QLV}^+$, we adopt the known canonical model method as in [8]. We, however, include the detailed outline to make this section self-contained.

Remark 1. We have two different axiomatizations of the set InqL_2 of all valid propositional formulas in inquisitive pair semantics. One proposed by Mascarenhas is the propositional intuitionistic logic IL extended with **W2**, **H2**, and atomic double negations ($\neg\neg\mathbf{p} \to \mathbf{p}$ for any atom $\mathbf{p}$). Another one proposed by Ciardelli and Roelofsen is IL extended with Kreisel-Putnam axiom **KP**: $(\neg A \to B \lor C) \to (\neg A \to B) \lor (\neg A \to C)$ and **H2**, and atomic double negations. And, if we drop **H2** from Ciardelli and Roelofsen's axiomatization, then we obtain the axiomatization of InqL, i.e., all valid propositional formulas in (generalized) inquisitive semantics. However, if we consider the first-order extension with **CD** of these logics, strong completeness of **IQL** extended with **CD** and **KP** for constant-domain Kripke semantics seems an open problem (p.c. by Valentin Shehtman and Silvio Ghilardi). Therefore, we choose Mascarenhas' axiomatization as a basis of our first-order inquisitive pair logic $\mathbf{QLV}^+$.

Let us expand our language $\mathcal{L}$ with a countable set $\{ \mathbf{c}_i \mid i \in \omega \}$ of *new* constant symbols. Let $\mathcal{L}^+$ be this expanded language of $\mathcal{L}$. We say that $\langle \Gamma; \Delta \rangle$ of $\mathcal{L}^+$ is *consistent* if $\nvdash \bigvee \Gamma_1 \to \bigwedge \Delta_1$ for any finite $\Gamma_1 \subseteq \Gamma$ and any finite $\Delta_1 \subseteq \Delta$. $\langle \Gamma; \Delta \rangle$ of $\mathcal{L}^+$ is *maximal* if $A \in \Gamma$ or $A \in \Delta$ for any formula A. $\langle \Gamma; \Delta \rangle$ of $\mathcal{L}^+$ is $\exists\forall$-*maximally consistent* if it is consistent and maximal and satisfies the following: ($L\exists$-**property**): For any formula of the form $\exists x. A$, if $\exists x. A \in \Gamma$, then $A[\mathbf{c}/x] \in \Gamma$ for some $\mathbf{c}$, and ($R\forall$-**property**): For any formula of the form $\forall x. A$, if $\forall x. A \in \Delta$, then $A[\mathbf{c}/x] \in \Delta$ for some $\mathbf{c}$. By consistency and maximality, it is obvious that $\Delta = \Gamma^c$, the complement of Γ [3]. So, if $\langle \Gamma; \Delta \rangle$ is $\exists\forall$-maximally consistent, then we usually say that Γ is an $\exists\forall$-*MCS*.

[3] Remark that we can easily derive from the consistency of $\langle \Gamma; \Delta \rangle$ that $\Gamma \cap \Delta = \emptyset$.

Lemma 1. (i) *If $\langle\, \Gamma \cup \{\, \exists\, x.\, A\,\}; \Delta\,\rangle$ is consistent and* $\mathbf{c}$ *does not occur in it, then $\langle\, \Gamma \cup \{\, \exists\, x.\, A, A[\mathbf{c}/x]\,\}; \Delta\,\rangle$ is consistent.* (ii) *If $\langle\, \Gamma; \Delta \cup \{\, \forall\, x.\, A\,\}\rangle$ is consistent and* $\mathbf{c}$ *does not occur in it, then $\langle\, \Gamma; \Delta \cup \{\, \forall\, x.\, A, A[\mathbf{c}/x]\,\}\,\rangle$ is consistent.* (iii) *If $\langle\, \Gamma; \Delta\,\rangle$ is consistent, then either $\langle\, \Gamma \cup \{\, A\,\}; \Delta\,\rangle$ or $\langle\, \Gamma; \Delta \cup \{\, A\,\}\,\rangle$ is consistent.*

Proof. We only establish (ii), since we need **CD** here. Suppose for contradiction that there exists some $\Gamma' \subseteq \Gamma$ and some $\Delta' \subseteq \Delta$ such that $\vdash \bigwedge \Gamma' \to \bigvee \Delta' \vee \forall\, x.\, A \vee A[\mathbf{c}/x]$. Fix some fresh y in $\langle\, \Gamma; \Delta \cup \{\, \forall\, x.\, A\,\}\rangle$. It is clear that $(A[y/x])[\mathbf{c}/y] \equiv A[\mathbf{c}/x]$. Since y and $\mathbf{c}$ are fresh, we obtain: $\vdash \bigwedge \Gamma' \to \forall\, y.\, (\bigvee \Delta' \vee \forall\, x.\, A \vee A[y/x])$. We deduce from **CD** that $\vdash \bigwedge \Gamma' \to (\bigvee \Delta' \vee \forall\, x.\, A)$ (remark that $\forall\, x.\, A$ and $\forall\, y.\, (A[y/x])$ are equivalent), which gives us the desired contradiction. □

Lemma 2. *If $\langle\, \Gamma; \Delta\,\rangle$ of $\mathcal{L}$ is consistent, then there exists $\langle\, \Gamma^+; \Delta^+\,\rangle$ of $\mathcal{L}^+$ such that $\Gamma \subseteq \Gamma^+$, $\Delta \subseteq \Delta^+$, and Γ^+ is an $\exists\forall$-MCS.*

Proof. Let us enumerate all the formulas of $\mathcal{L}^+$ as $(F_n)_{n\in\omega}$. Recall that all the new constant symbols $\{\, \mathbf{c}_i \mid i \in \omega\,\}$ are indexed by $i \in \omega$. In what follows, we define a sequence $(\langle\, \Gamma_n; \Delta_n\,\rangle)_{n\in\omega}$ such that each $\langle\, \Gamma_n; \Delta_n\,\rangle$ is consistent, and obtain $\langle\, \Gamma^+; \Delta^+\,\rangle := \langle\, \bigcup_{n\in\omega} \Gamma_n; \bigcup_{n\in\omega} \Delta_n\,\rangle$ as its limit. (Basis) Put $\Gamma_0 := \Gamma$ and $\Delta_0 := \Delta$. (Inductive Step) Suppose that we have defined a consistent $\langle\, \Gamma_n; \Delta_n\,\rangle$. We subdivide our argument into the following three cases: (a) $F_n \equiv \exists\, x.\, A$ and $\langle\, \Gamma_n \cup \{\, F_n\,\}; \Delta_n\,\rangle$ is consistent; (b) $F_n \equiv \forall\, x.\, A$ and $\langle\, \Gamma_n; \Delta_n \cup \{\, F_n\,\}\rangle$ is consistent; (c) Otherwise. First, we show the case (c). Since either $\langle\, \Gamma_n \cup \{\, F_n\,\}; \Delta_n\,\rangle$ or $\langle\, \Gamma_n; \Delta_n \cup \{\, F_n\,\}\rangle$ is consistent by Lemma 1 (iii), choose a consistent pair and define it as $\langle\, \Gamma_{n+1}, \Delta_{n+1}\,\rangle$. As for the case (a), let us choose a fresh $\mathbf{c}$ in $\langle\, \Gamma_n \cup \{\, F_n\,\}; \Delta_n\,\rangle$ by Lemma 1 (i) and define $\langle\, \Gamma_{n+1}, \Delta_{n+1}\,\rangle := \langle\, \Gamma_n \cup \{\, \exists\, x.\, A, A[\mathbf{c}/x]\,\}; \Delta_n\,\rangle$. As for the case (b) (similarly to (a)), let us choose a fresh $\mathbf{c}$ in $\langle\, \Gamma_n; \Delta_n \cup \{\, F_n\,\}\rangle$ by Lemma 1 (ii) and define $\langle\, \Gamma_{n+1}, \Delta_{n+1}\,\rangle := \langle\, \Gamma_n; \Delta_n \cup \{\, \forall\, x.\, A, A[\mathbf{c}/x]\,\}\rangle$.

Finally, it is easy to see that $\langle\, \bigcup_{n\in\omega} \Gamma_n; \bigcup_{n\in\omega} \Delta_n\,\rangle$ is $\exists\forall$-maximally consistent. □

Γ is *ω-closed* if, for any formula of the form $\forall\, x.\, A$ in $\mathcal{L}^+$, if $\Gamma \vdash A[\mathbf{c}/x]$ for all constants $\mathbf{c}$ then $\Gamma \vdash \forall\, x.\, A$. $\langle\, \Gamma; \Delta\,\rangle$ is *ω-closed-finite-consistent* (in short, ωfc) if Γ is ω-closed and Δ is finite and $\langle\, \Gamma; \Delta\,\rangle$ is consistent. We can easily show the following:

Lemma 3. *If Γ is an $\exists\forall$-MCS, then Γ is ω-closed.*

Lemma 4. (i) *If Γ is ω-closed, then $\Gamma \cup \{\, A\,\}$ is also ω-closed.* (ii) *If $\langle\, \Gamma \cup \{\, \exists\, x.\, A\,\}; \Delta\,\rangle$ is ωfc, then there exists some* $\mathbf{c}$ *such that $\langle\, \Gamma \cup \{\, \exists\, x.\, A, A[\mathbf{c}/x]\,\}; \Delta\,\rangle$ is consistent.* (iii) *If $\langle\, \Gamma; \Delta \cup \{\, \forall\, x.\, A\,\}\rangle$ is ωfc there exists some* $\mathbf{c}$ *such that $\langle\, \Gamma; \Delta \cup \{\, \forall\, x.\, A, A[\mathbf{c}/x]\,\}\rangle$ is consistent.*

Proof. We only establish (iii), since we need **CD** here. Suppose that $\langle\, \Gamma; \Delta \cup \{\, \forall\, x.\, A\,\}\rangle$ is ωfc. Assume for contradiction that $\langle\, \Gamma; \Delta \cup \{\, \forall\, x.\, A, A[\mathbf{c}/x]\,\}\rangle$ is inconsistent for all constant symbol $\mathbf{c}$. By finiteness of Δ, we can assume w.l.o.g. that x does not occur in Δ (otherwise, it suffices to rename the bounded variable). Then, for all constant $\mathbf{c}$, we have $\Gamma \vdash \bigvee \Delta \vee (\forall\, x.\, A) \vee A[\mathbf{c}/x]$, i.e., $\Gamma \vdash (\bigvee \Delta \vee (\forall\, x.\, A) \vee A)[\mathbf{c}/x]$. Since Γ is ω-closed, $\Gamma \vdash \forall\, x.\, (\bigvee \Delta \vee (\forall\, x.\, A) \vee A)$. By **CD**, $\vdash \forall\, x.\, (\bigvee \Delta \vee (\forall\, x.\, A) \vee A) \to \bigvee \Delta \vee (\forall\, x.\, A)$. Therefore, we get $\Gamma \vdash \bigvee \Delta \vee (\forall\, x.\, A)$, which contradicts the consistency of $\langle\, \Gamma; \Delta \cup \{\, \forall\, x.\, A\,\}\rangle$. □

Lemma 5. *If $\langle \Gamma; \Delta \rangle$ of $\mathcal{L}^+$ is ωfc, then there exists $\langle \Gamma^+; \Delta^+ \rangle$ of $\mathcal{L}^+$ such that $\Gamma \subseteq \Gamma^+$, $\Delta \subseteq \Delta^+$, and Γ^+ is an $\exists\forall$-MCS.*

Proof. The proof is similar to the proof of Lemma 2. We, however, need to care about the fact that $\langle \Gamma; \Delta \rangle$ is ωfc. Fix any enumeration $(F_n)_{n\in\omega}$ of all the formulas of $\mathcal{L}^+$. In what follows, we only describe the difference from the proof of Lemma 2. Below, we define a sequence $(\langle \Gamma_n; \Delta_n \rangle)_{n\in\omega}$ such that *each $\langle \Gamma_n; \Delta_n \rangle$ is ωfc*, and obtain $\langle \Gamma^+; \Delta^+ \rangle$:= $\langle \bigcup_{n\in\omega} \Gamma_n; \bigcup_{n\in\omega} \Delta_n \rangle$. The basis step is the same as before. As for the inductive step, suppose that we have defined an ωfc $\langle \Gamma_n; \Delta_n \rangle$. We subdivide our argument into the cases (a), (b), and (c) in the same way as in the proof of Lemma 2. The definition of $\langle \Gamma_{n+1}; \Delta_{n+1} \rangle$ for each case is exactly the same as before. However, we need to check that we can find some constant $\mathbf{c}$ in both the cases (a) and (b) (the most important point is: there is no need for $\mathbf{c}$ to be *fresh*) and that $\langle \Gamma_{n+1}; \Delta_{n+1} \rangle$ is also ωfc. We can ensure these points by Lemma 4. $\square$

Lemma 6. *Let Γ be an $\exists\forall$-MCS. Then:* (i) $A \wedge B \in \Gamma$ iff ($A \in \Gamma$ *and* $B \in \Gamma$), (ii) $A \vee B \in \Gamma$ iff ($A \in \Gamma$ *or* $B \in \Gamma$), (iii) $\forall x. A \in \Gamma$ iff $A[t/x] \in \Gamma$ *for any term t,* (iv) $\exists x. A \in \Gamma$ iff $A[t/x] \in \Gamma$ *for some term t,* (v) *If $A \to B \in \Gamma$ and $A \in \Gamma$, then $B \in \Gamma$,* (vi) ($\neg A \in \Gamma$ *and* $A \in \Gamma$) *fails.*

Proof. Assume that $\langle \Gamma; \Delta \rangle$ is $\exists\forall$-maximally consistent. We only show (iii). By $\vdash \forall x. A \to A[t/x]$, we can establish the left-to-right direction. As for the right-to-left direction, assume $\forall x. A \notin \Gamma$. By maximality, $\forall x. A \in \Delta$. By $R\forall$-property, $A[\mathbf{c}/x] \in \Delta$ for some constant $\mathbf{c}$. So, there exists a term t such that $A[t/x] \notin \Gamma$ by the consistency. $\square$

Definition 5. *The* canonical model for $\mathbf{QLV}^+$ $\mathfrak{M} = \langle W, \leq, D, V \rangle$ *is defined by:* (i) $W = \{ \Gamma \mid \Gamma \text{ is an } \exists\forall\text{-MCS} \}$ [4], (ii) $\Gamma \leq \Pi$ iff $\Gamma \subseteq \Pi$, (iii) $D = \{ t \mid t \text{ is a term of } \mathcal{L}^+ \}$, (vi) $c^V = c$ *for any constant symbol c in $\mathcal{L}^+$,* (v) $\langle t_1, \ldots, t_n \rangle \in P^V_\Gamma$ iff $P(t_1, \ldots, t_n) \in \Gamma$.

Lemma 7 (Truth Lemma). *Let $\mathfrak{M} = \langle W, \leq, D, V \rangle$ be the canonical model for* $\mathbf{QLV}^+$. *Define the canonical assignment g by $g(x) = x$. Then, $\mathfrak{M}, \Gamma, g \Vdash A$ iff $A \in \Gamma$.*

Proof. By induction on A. First, let us remark that $\overline{g}(t) = t$ for any term t of $\mathcal{L}^+$. By Lemma 6 and the definition of the canonical model, we can easily establish the cases where $A \equiv P(t_1, \cdots, t_n)$, $B \vee C$, $B \wedge C$, $\exists x. B$ or $\forall x. B$ (if $A \equiv \exists x. B$ or $\forall x. B$, we need to use: $\mathfrak{M}, \Gamma, g(x|t) \Vdash A$ iff $\mathfrak{M}, \Gamma, g \Vdash A[t/x]$). So, let us only show the case where $A \equiv B \to C$. In order to establish the left-to-right direction, assume $B \to C \notin \Gamma$. By maximality, $B \to C \in \Delta$, where $\Delta = \Gamma^c$. By consistency of $\langle \Gamma; \Delta \rangle$, $\langle \Gamma \cup \{ B \}; \{ C \} \rangle$ is consistent. By Lemma 3 and Lemma 4 (i), $\langle \Gamma \cup \{ B \}; \{ C \} \rangle$ is ωfc. It follows from Lemma 5 that there exists some $\langle \Gamma^+; \Delta^+ \rangle$ such that Γ^+ is an $\exists\forall$-MCS and $\Gamma \cup \{ B \} \subseteq \Gamma^+$ and $C \in \Delta^+$ (i.e., $C \notin \Gamma^+$ by the consistency). By the induction hypothesis, we obtain: $\mathfrak{M}, \Gamma, g \Vdash B$ and $\mathfrak{M}, \Gamma, g \nVdash C$. Since $\Gamma \subseteq \Gamma^+$, we conclude that $\mathfrak{M}, \Gamma, g \nVdash B \to C$. Finally, let us show the right-to-left direction. Assume $\mathfrak{M}, \Gamma, g \nVdash B \to C$, i.e., there exists some $\exists\forall$-MCS Γ' such that $\mathfrak{M}, \Gamma', g \Vdash B$ and $\mathfrak{M}, \Gamma', g \nVdash C$. By the induction hypothesis, we obtain: $B \in \Gamma'$ and $C \notin \Gamma'$. It follows from Lemma 6 (v) that $B \to C \notin \Gamma'$. $\square$

[4] Remark that any MCS Γ is a $\mathbf{QLV}^+$-theory. This is shown as follows: Given any MCS Γ, assume that $\varphi \in \Gamma$ and $\varphi \vdash \psi$. Suppose for contradiction that $\psi \notin \Gamma$. By maximality, $\psi \in \Delta$. By consistency, we get $\nvdash \varphi \to \psi$, which contradicts $\varphi \vdash \psi$.

Lemma 8. *Let $\mathfrak{M} = \langle W, \leq, D, V \rangle$ be the canonical model for* $\mathbf{QLV}^+$. *Then,* (i) $\mathfrak{M}$ *satisfies* ($h2$)*,* (ii) $\mathfrak{M}$ *satisfies* ($w2$)*,* (iii) $\mathfrak{M}$ *has the intersection property.*

Proof. We can show (i) and (ii) in the same way as in the propositional case [2, Theorem 35] (for (i), the reader can also refer to [8, Lemma 7.3.3 (1)]). So, we only show (iii). Let Γ be an $\exists\forall$-MCS. It suffices to show that: $P(t_1, \ldots, t_n) \in \Gamma$ iff $P(t_1, \ldots, t_n) \in \bigcap \{ \Gamma' \mid \Gamma \subseteq \Gamma'$ and Γ' is an endpoint$\}$ (remark that ($w2$) and ($h2$) assure us that, for any Γ in $\mathfrak{M}$, there exists some endpoint $\Gamma' \supseteq \Gamma$). We can easily show the left-to-right direction. So, let us establish the right-to-left direction. Assume that $P(t_1, \ldots, t_n) \in \Gamma'$ for any $\Gamma' \supseteq \Gamma$ such that Γ' is an endpoint. By ($w2$) and ($h2$), we can state that, for any state $\Pi \supseteq \Gamma$, there exists an endpoint $\Theta \supseteq \Pi$. Thus, we deduce from Truth Lemma that $\mathfrak{M}, \Gamma, g \Vdash \neg\neg P(t_1, \ldots, t_n)$, i.e., $\neg\neg P(t_1, \ldots, t_n) \in \Gamma$. Since $\vdash \neg\neg P(t_1, \cdots, t_n) \to P(t_1, \ldots, t_n)$, we can conclude that $P(t_1, \ldots, t_n) \in \Gamma$. ◻

Theorem 2. $\Gamma \Vdash_{\mathsf{VI}} A$ iff $\Gamma \vdash A$.

Proof. We can easily show that $\Gamma \vdash A$ implies $\Gamma \Vdash_{\mathsf{VI}} A$. So, let us establish the left-to-right direction. We show the contrapositive implication. Assume $\Gamma \nvdash A$ (remark that Γ might be infinite). Then, $\langle \Gamma, A \rangle$ is consistent. By Lemma 2, there exists some $\langle \Gamma^+; \Delta^+ \rangle$ such that $\Gamma \subseteq \Gamma^+$, $A \in \Delta^+$, and Γ^+ is an $\exists\forall$-MCS. By consistency of $\langle \Gamma^+; \Delta^+ \rangle$, $A \notin \Gamma^+$. It follows from Truth Lemma that $\mathfrak{M}, \Gamma^+, g \Vdash B$ ($B \in \Gamma$) and $\mathfrak{M}, \Gamma^+, g \nVdash A$. By Lemma 8, $\Gamma \nVdash_{\mathsf{VI}} A$, as desired. ◻

Corollary 1. *The following are all equivalent:* (i) $\Gamma \Vdash_{\mathsf{VI}} A$; (ii) $\Gamma \models_{\mathsf{M}_{\mathrm{all}}} A$; (iii) $\Gamma \models_{\mathsf{M}_{\mathrm{int}}} A$; (iv) $\Gamma \vdash A$.

Proof. Theorem 1 gives us the equivalence among (i), (ii), and (iii). Theorem 2 ensures the equivalence between (i) and (iv). ◻

4 Tree-Sequent Calculus for First-Order Inquisitive Pair Logic

In this section, we first introduce a tree-sequent calculus for $\mathsf{InqQL}_2 = \{ A \mid \mathsf{M}_{\mathrm{all}} \models A \}$, as a special form of Labelled Deductive Systems [11].

Let $\mathcal{T} = \langle \{0, 1, 2\}, \leq \rangle$ be the tree equipped with the order $\leq := \{ \langle 0, 1 \rangle, \langle 0, 2 \rangle \} \cup \{ \langle x, x \rangle \mid x \in \{0, 1, 2\} \}$. A *label* is an element of $\{0, 1, 2\}$. We use letters α, β, etc. for labels. A *labelled formula* is a pair $\alpha : A$, where α is a label and A is a formula of the language $\mathcal{L}$. In what follows in this paper, we use Γ, Δ, etc. to denote a set of *labelled formulas*. A *tree-sequent* is an expression $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of labelled formulas.

Now, let us introduce the tree-sequent calculus $\mathbf{T}\mathsf{InqQL}_2$ for first-order inquisitive pair logic InqQL_2. This system defines inference schemes which allow us to manipulate tree-sequents. The axioms of $\mathbf{T}\mathsf{InqQL}_2$ are of the following forms:

$$\alpha : A, \Gamma \Rightarrow \Delta, \alpha : A \ (\mathrm{Ax}) \qquad \alpha : \bot, \Gamma \Rightarrow \Delta \ (\bot \mathrm{L}).$$

The inference rules of $\mathbf{T}\mathsf{InqQL}_2$ are the following:

$$\frac{0 : P(t_1, \ldots, t_n), \Gamma \Rightarrow \Delta}{1 : P(t_1, \ldots, t_n), 2 : P(t_1, \ldots, t_n), \Gamma \Rightarrow \Delta} \ (\text{Atom L}) \qquad \frac{1 : A, 2 : A, \Gamma \Rightarrow \Delta}{0 : A, \Gamma \Rightarrow \Delta} \ (\text{Move})$$

$$\frac{\alpha : A, \alpha : B, \Gamma \Rightarrow \Delta}{\alpha : A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\mathrm{L}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha : A \quad \Gamma \Rightarrow \Delta, \alpha : B}{\Gamma \Rightarrow \Delta, \alpha : A \wedge B}\ (\wedge\mathrm{R})$$

$$\frac{\alpha : A, \Gamma \Rightarrow \Delta \quad \alpha : B, \Gamma \Rightarrow \Delta}{\alpha : A \vee B, \Gamma \Rightarrow \Delta}\ (\vee\mathrm{L}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha : A, \alpha : B}{\Gamma \Rightarrow \Delta, \alpha : A \vee B}\ (\vee\mathrm{R})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha : A}{\alpha : \neg A, \Gamma \Rightarrow \Delta}\ (\neg\mathrm{L}) \qquad \frac{\alpha : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha : \neg A}\ (\neg\mathrm{R}_{1,2}) \text{ where } \alpha \neq 0$$

$$\frac{1 : A, \Gamma \Rightarrow \Delta \quad 2 : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, 0 : \neg A}\ (\neg\mathrm{R}_0)$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha : A \quad \alpha : B, \Gamma \Rightarrow \Delta}{\alpha : A \to B, \Gamma \Rightarrow \Delta}\ (\to\mathrm{L}) \qquad \frac{\alpha : A, \Gamma \Rightarrow \Delta, \alpha : B}{\Gamma \Rightarrow \Delta, \alpha : A \to B}\ (\to\mathrm{R}_{1,2}) \text{ where } \alpha \neq 0$$

$$\frac{0 : A, \Gamma \Rightarrow \Delta, 0 : B \quad 1 : A, \Gamma \Rightarrow \Delta, 1 : B \quad 2 : A, \Gamma \Rightarrow \Delta, 2 : B}{\Gamma \Rightarrow \Delta, 0 : A \to B}\ (\to\mathrm{R}_0)$$

$$\frac{\alpha : A[t/x], \Gamma \Rightarrow \Delta}{\alpha : \forall x.\, A, \Gamma \Rightarrow \Delta}\ (\forall L) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha : A[z/x]}{\Gamma \Rightarrow \Delta, \alpha : \forall x.\, A}\ (\forall R)\dagger$$

$$\frac{\alpha : A[z/x], \Gamma \Rightarrow \Delta}{\alpha : \exists x.\, A, \Gamma \Rightarrow \Delta}\ (\exists L)\dagger \qquad \frac{\Gamma \Rightarrow \Delta, \alpha : A[t/x]}{\alpha : \Gamma \Rightarrow \Delta, \exists x.\, A}\ (\exists R)$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha : A \quad \alpha : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{Cut})$$

where † means the *eigenvariable condition*: z does not occur in the conclusion. The tree-sequent calculus **cutfreeT**InqQL_2 is obtained by dropping (Cut) from **T**InqQL_2. Whenever a tree-sequent $\Gamma \Rightarrow \Delta$ is provable in **T**InqQL_2 (or, in **cutfreeT**InqQL_2), we write **T**$\mathsf{InqQL}_2 \vdash \Gamma \Rightarrow \Delta$ (or, **cutfreeT**$\mathsf{InqQL}_2 \vdash \Gamma \Rightarrow \Delta$, respectively).

4.1 Completeness of Tree-Sequent Calculus

In this subsection, we show that the tree-sequent calculus **cutfreeT**InqQL_2 is sufficient to prove all formulas that are valid in $\mathsf{M}_{\mathrm{all}}$.

In the following, Γ, Δ are possibly infinite in the expression $\Gamma \Rightarrow \Delta$ of a tree-sequent. In the case where Γ, Δ are all finite, the tree-sequent $\Gamma \Rightarrow \Delta$ said to be *finite*. A (possibly infinite) tree-sequent $\Gamma \Rightarrow \Delta$ is *provable* in **cutfreeT**InqQL_2, if **cutfreeT**$\mathsf{InqQL}_2 \vdash \Gamma' \Rightarrow \Delta'$ for some finite tree-sequent $\Gamma' \Rightarrow \Delta'$ such that $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$. In what follows, we extend our notation **cutfreeT**$\mathsf{InqQL}_2 \vdash \Gamma \Rightarrow \Delta$ to cover any possibly infinite tree-sequent in the sense explained above.

Definition 6. *A tree-sequent $\Gamma \Rightarrow \Delta$ is* saturated *if it satisfies the following:*

(consistency) (i) *If $\alpha : A \in \Gamma$, then $\alpha : A \notin \Delta$,* (ii) $\alpha : \bot \notin \Gamma$.
(persistence condition) *If $0 : A \in \Gamma$, then $1 : A \in \Gamma$ and $2 : A \in \Gamma$.*
(atom l) *If $1 : P(t_1, \ldots, t_n) \in \Gamma$ and $2 : P(t_1, \ldots, t_n) \in \Gamma$, then $0 : P(t_1, \ldots, t_n) \in \Gamma$.*
($\wedge$l) *If $\alpha : A \wedge B \in \Gamma$, then $\alpha : A \in \Gamma$ and $\alpha : B \in \Gamma$.*
($\wedge$r) *If $\alpha : A \wedge B \in \Delta$, then $\alpha : A \in \Delta$ or $\alpha : B \in \Delta$.*
($\vee$l) *If $\alpha : A \vee B \in \Gamma$, then $\alpha : A \in \Gamma$ or $\alpha : B \in \Gamma$.*
($\vee$r) *If $\alpha : A \vee B \in \Delta$, then $\alpha : A \in \Delta$ and $\alpha : B \in \Delta$.*
($\neg$l) *If $\alpha : \neg A \in \Gamma$, then $\alpha : A \in \Delta$.*
($\neg$r$_{1,2}$) *If $\alpha : \neg A \in \Delta$ and $\alpha \neq 0$, then $\alpha : A \in \Gamma$.*
($\neg$r$_0$) *If $0 : \neg A \in \Delta$, then $1 : A \in \Gamma$ or $2 : A \in \Gamma$.*
($\to$l) *If $\alpha : A \to B \in \Gamma$, then $\alpha : A \in \Delta$ or $\alpha : B \in \Gamma$.*
($\to$r$_{1,2}$) *If $\alpha : A \to B \in \Delta$ and $\alpha \neq 0$, then $\alpha : A \in \Gamma$ and $\alpha : B \in \Delta$.*

($\to$r$_0$) *If* $0 : A \to B \in \varDelta$, *then* ($\alpha : A \in \varGamma$ *and* $\alpha : B \in \varDelta$) *for some* $\alpha \in \{0, 1, 2\}$.
($\forall$l) *If* $\alpha : \forall x. A \in \varGamma$, *then* $\alpha : A[t/x] \in \varGamma$ *for any term* t.
($\forall$r) *If* $\alpha : \forall x. A \in \varDelta$, *then* $\alpha : A[z/x] \in \varDelta$ *for some variable* z.
($\exists$l) *If* $\alpha : \exists x. A \in \varGamma$, *then* $\alpha : A[z/x] \in \varGamma$ *for some variable* z.
($\exists$r) *If* $\alpha : \exists x. A \in \varDelta$, *then* $\alpha : A[t/x] \in \varDelta$ *for any term* t.

Lemma 9. *If a finite tree-sequent* $\varGamma \Rightarrow \varDelta$ *is not provable in* **cutfreeT**InqQL_2, *then there exists a saturated tree-sequent* $\varGamma^+ \Rightarrow \varDelta^+$ *such that* $\varGamma \subseteq \varGamma^+$ *and* $\varDelta \subseteq \varDelta^+$ *and* $\varGamma^+ \Rightarrow \varDelta^+$ *is not provable in* **cutfreeT**InqQL_2.

The proof of this lemma can be found in Appendix A. Each node α of a tree-sequent $\varGamma \Rightarrow \varDelta$ is associated with a sequent $\varGamma_\alpha \Rightarrow \varDelta_\alpha$ where $\varGamma_\alpha$ (or, $\varDelta_\alpha$) is the set of formulas such that $\alpha : A \in \varGamma$ (or, $\alpha : A \in \varDelta$, respectively). We define a translation of tree-sequents into formulas of $\mathcal{L}$. In the following definition, tree-sequents are all finite. Let $\varGamma \Rightarrow \varDelta$ be a tree-sequent and **s**, **t** be fresh *propositional variables* in $\varGamma \Rightarrow \varDelta$. The *formulaic translation* $[\![\varGamma \Rightarrow \varDelta]\!]$ is defined as (note that the following formulaic translation depends on the choice of **s** and **t**):

$$[\![\varGamma \Rightarrow \varDelta]\!] \equiv \bigwedge \varGamma_0 \to ((\mathbf{s} \vee \mathbf{t}) \vee \bigvee \varDelta_0 \vee [\![\varGamma \Rightarrow \varDelta]\!]_1 \vee [\![\varGamma \Rightarrow \varDelta]\!]_2) \text{ where:}$$
$$[\![\varGamma \Rightarrow \varDelta]\!]_1 \equiv \mathbf{s} \wedge \bigwedge \varGamma_1 \to \mathbf{t} \vee \bigvee \varDelta_1; \qquad [\![\varGamma \Rightarrow \varDelta]\!]_2 \equiv \mathbf{t} \wedge \bigwedge \varGamma_2 \to \mathbf{s} \vee \bigvee \varDelta_2.$$

An idea behind fresh **s** and **t** is to *name* three pairwise subsets (corresponding to 0, 1, 2 in our fixed tree) in an inquisitive model. Recall that $\mathsf{M}_{\mathrm{int}}$ is the class of all *intended* inquisitive models.

Theorem 3. *If* $\mathsf{M}_{\mathrm{int}} \models [\![\varGamma \Rightarrow \varDelta]\!]$, *then* **cutfreeT**$\mathsf{InqQL}_2 \vdash \varGamma \Rightarrow \varDelta$. *Therefore, if* $\mathsf{M}_{\mathrm{all}} \models [\![\varGamma \Rightarrow \varDelta]\!]$, *then* **cutfreeT**$\mathsf{InqQL}_2 \vdash \varGamma \Rightarrow \varDelta$.

Proof. It suffices to establish the first part. We show the contrapositive implication of it. Assume that $\varGamma \Rightarrow \varDelta$ is unprovable in **cutfreeT**InqQL_2. Then, by Lemma 9, there exists some saturated tree-sequent $\varGamma^+ \Rightarrow \varDelta^+$ such that $0 : A \in \varDelta^+$ and **cutfreeT**$\mathsf{InqQL}_2 \nvdash \varGamma^+ \Rightarrow \varDelta^+$. Define $D = \{t \mid t \text{ is a term of } \mathcal{L}\}$. We define an interpretation $\mathscr{I}$ of constant symbols on D by $\mathscr{I}(c) := c$ and an assignment g by $g(x) = x$. Let us define the following two first-order classical structure $\mathfrak{A}_1$ and $\mathfrak{A}_2$: $|\mathfrak{A}_1| = |\mathfrak{A}_2| = D$, $c^{\mathfrak{A}_1} = c^{\mathfrak{A}_2} = \mathscr{I}(c)$, $P^{\mathfrak{A}_1} = \{\langle t_1, \dots, t_n \rangle \mid 1 : P(t_1, \dots, t_n) \in \varGamma^+\}$, $P^{\mathfrak{A}_2} = \{\langle t_1, \dots, t_n \rangle \mid 2 : P(t_1, \dots, t_n) \in \varGamma^+\}$. Now we show by induction on X of $\mathcal{L}$ that:

- (i) If $0 : X \in \varGamma^+$ then $\{\mathfrak{A}_1, \mathfrak{A}_2\}, g \models X$; (ii) If $0 : X \in \varDelta^+$ then $\{\mathfrak{A}_1, \mathfrak{A}_2\}, g \not\models X$.
- (iii) If $\alpha : X \in \varGamma^+$ and $\alpha \neq 0$, then $\{\mathfrak{A}_\alpha\}, g \models X$; (iv) If $\alpha : X \in \varDelta^+$ and $\alpha \neq 0$, then $\{\mathfrak{A}_\alpha\}, g \not\models X$.

Here we consider only the cases where X is of the form $P(t_1, \dots, t_n)$ and of the form $\forall x. B$ (for the cases X is of the form $\neg B$ and of the form $B \to C$, the reader can find an essential argument in the proof of [4, Theorem 1]).
(The case where X is of the form $P(t_1, \dots, t_n)$) We only show the cases (i) and (ii). (i) Suppose that $0 : P(t_1, \dots, t_n) \in \varGamma^+$. Since $\varGamma^+ \Rightarrow \varDelta^+$ is saturated, $1 : P(t_1, \dots, t_n), 2 : P(t_1, \dots, t_n) \in \varGamma^+ \in \varGamma^+$ by (**persistence condition**). So, $\langle t_1, \dots, t_n \rangle \in P^{\mathfrak{A}_1}$ and $\langle t_1, \dots, t_n \rangle \in P^{\mathfrak{A}_2}$. Since $\overline{g}(t) = t$, we can deduce that $\{\mathfrak{A}_1, \mathfrak{A}_2\}, g \models P(t_1, \dots, t_n)$. (ii) Suppose that

$0 : P(t_1, \ldots, t_n) \in \varDelta^+$. Since **cutfreeT**$\mathsf{InqQL}_2 \nvdash \varGamma^+ \Rightarrow \varDelta^+$ and $\varGamma^+ \Rightarrow \varDelta^+$ is saturated, $0 : P(t_1, \ldots, t_n) \notin \varGamma^+$ by (**consistency**). $0 : P(t_1, \ldots, t_n) \notin \varGamma^+$ means that $1 : P(t_1, \ldots, t_n) \notin \varGamma^+$ or $2 : P(t_1, \ldots, t_n) \notin \varGamma^+$ by (**atoml**). So, $\langle t_1, \ldots, t_n \rangle \notin P^{\mathfrak{A}_1}$ or $\langle t_1, \ldots, t_n \rangle \notin P^{\mathfrak{A}_2}$. Therefore, by $\overline{g}(t) = t$, $\{\mathfrak{A}_1, \mathfrak{A}_2\}, g \models P(t_1, \ldots, t_n)$, as desired.
(The case where X is of the form $\forall x. B$) We only show the cases (i) and (ii). (i) Suppose that $0 : \forall x. B \in \varGamma^+$. Since $\varGamma^+ \Rightarrow \varDelta^+$ is saturated, $0 : B[t/x] \in \varGamma^+$ for any term t by ($\forall$**l**). By the induction hypothesis, we have: for any term t, $\{\mathfrak{A}_1, \mathfrak{A}_2\}, g \models B[t/x]$, i.e., $\{\mathfrak{A}_1, \mathfrak{A}_2\}, g(x|t) \models B$. Therefore, $\{\mathfrak{A}_1, \mathfrak{A}_2\}, g \models \forall x. B$. (ii) Suppose that $0 : \forall x. B \in \varDelta^+$. Since $\varGamma^+ \Rightarrow \varDelta^+$ is saturated, $0 : B[z/x] \in \varDelta^+$ for any some variable z by ($\forall$**r**). By the induction hypothesis, we have: for some variable z, $\{\mathfrak{A}_1, \mathfrak{A}_2\}, g \models B[z/x]$, i.e., $\{\mathfrak{A}_1, \mathfrak{A}_2\}, g(x|z) \models B$. Therefore, $\{\mathfrak{A}_1, \mathfrak{A}_2\}, g \not\models \forall x. B$.

Let us choose fresh **s** and **t** in $\varGamma^+ \Rightarrow \varDelta^+$ for $[\![\varGamma \Rightarrow \varDelta]\!]$ and expand our model above so that **s** is true only under $\mathfrak{A}_1$ and **t** is true only under $\mathfrak{A}_2$. Then, we can conclude that $[\![\varGamma \Rightarrow \varDelta]\!]$ is not valid in $\mathsf{M}_{\mathrm{int}}$ by construction of our model and (i) - (iv) above. □

4.2 Cut-Elimination Theorem and Soundness of Tree-Sequent Calculus

In this subsection, we establish that the tree-sequent calculus **T**InqQL_2 (i.e., **cutfreeT**InqQL_2 with (Cut)) enjoys a cut-elimination theorem and that it is sound with respect to the class $\mathsf{M}_{\mathrm{all}}$ of all inquisitive models.

Lemma 10. *If* **T**$\mathsf{InqQL}_2 \vdash \varGamma \Rightarrow \varDelta$*, then* $\mathsf{M}_{\mathrm{all}} \models [\![\varGamma \Rightarrow \varDelta]\!]$.

The proof of this lemma can be found in Appendix B.

Theorem 4. *If* **T**$\mathsf{InqQL}_2 \vdash \varGamma \Rightarrow \varDelta$*, then* **cutfreeT**$\mathsf{InqQL}_2 \vdash \varGamma \Rightarrow \varDelta$.

Proof. It follows from Lemma 10 and Theorem 3. □

In order to establish the soundness through our formulaic translation with fresh variables, we need to show the following, which lets us use the fresh propositional variables **s** and **t** to *name* three pairwise subsets (corresponding to 0, 1, 2 in our fixed tree) in an inquisitive model.

Lemma 11. *If* $\mathsf{M}_{\mathrm{all}} \models (\mathbf{s} \vee \mathbf{t}) \vee A \vee (\mathbf{s} \to \mathbf{t}) \vee (\mathbf{t} \to \mathbf{s})$*, then* $\mathsf{M}_{\mathrm{all}} \models A$*, where* **s** *and* **t** *are fresh in* A.

Proof. Assume $\mathsf{M}_{\mathrm{all}} \not\models A$. By Proposition 3 (iv), there exists some inquisitive model $\mathfrak{M} = \langle W, D, V \rangle$, some $w, v \in W$ and some assignment g such that $w \neq v$ and $\#W \geq 2$ and $\{w, v\}, g \not\models_{\mathfrak{M}} A$. Let V' be the same valuation as V except that **s** is true only at w and **t** is true only at v under V'. Write $\mathfrak{M}' = \langle W, D, V' \rangle$. Then, $s, g \models_{\mathfrak{M}} B$ iff $s, g \models_{\mathfrak{M}'} B$, for any $s \subseteq \{w, v\}$ and any subformula B of A. Thus, $\{w, v\}, g \not\models_{\mathfrak{M}'} A$. By definition of V', $\{w, v\}, g \not\models_{\mathfrak{M}'} (\mathbf{s} \vee \mathbf{t}) \vee A \vee (\mathbf{s} \to \mathbf{t}) \vee (\mathbf{t} \to \mathbf{s})$, as required. □

Theorem 5. *If* **T**$\mathsf{InqQL}_2 \vdash\Rightarrow 0 : A$*, then* $\mathsf{M}_{\mathrm{all}} \models A$.

Proof. By Lemma 10, $[\![\Rightarrow 0 : A]\!]$ is valid in $\mathsf{M}_{\mathrm{all}}$, i.e., $(\mathbf{s} \vee \mathbf{t}) \vee A \vee (\mathbf{s} \to \mathbf{t}) \vee (\mathbf{t} \to \mathbf{s})$ is valid in $\mathsf{M}_{\mathrm{all}}$. It follows from Lemma 11 that A is valid in $\mathsf{M}_{\mathrm{all}}$. □

5 Conclusion

Corollary 2. *The following are equivalent:* (i) **cutfree**$\mathbf{T}\mathsf{InqQL}_2 \vdash\Rightarrow 0 : A$; (ii) $\mathbf{T}\mathsf{InqQL}_2 \vdash\Rightarrow 0 : A$; (iii) $\mathsf{M}_{\mathrm{all}} \models A$; (iv) $\mathsf{M}_{\mathrm{int}} \models A$; (v) $\mathsf{VI} \Vdash A$; (vi) $\mathbf{QLV}^+ \vdash A$.

Proof. By Corollary 1, we establish the equivalence among (iii), (iv), (v) and (vi) (put $\Gamma = \emptyset$). By Theorem 3, (iii) $\Rightarrow$ (i). Trivially, (i) $\Rightarrow$ (ii). By Theorem 5, (ii) $\Rightarrow$ (iii). $\square$

Our proof process for Corollary 2 also reveals that $\mathbf{T}\mathsf{InqQL}_2$ corresponds to $\mathbf{QLV}^+$ extended with the following non-standard proof rule: From $(\mathbf{s} \vee \mathbf{t}) \vee A \vee (\mathbf{s} \to \mathbf{t}) \vee (\mathbf{t} \to \mathbf{s})$, we may infer A, where $\mathbf{s}$ and $\mathbf{t}$ are fresh propositional variables in A. One of the main causes of such logical phenomena consists in the fact that we use the fixed tree $\mathcal{T}$, unlike the previous studies [12,9] which employ 'growing' tree-sequents. Therefore, this study also contributes to witness a logical connection between labelled deductive systems with a *fixed set of labels* and Hilbert-style axiomatizations with *non-standard proof-rules* [5] [6].

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[5] We can reduce the completeness of $\mathbf{T}\mathsf{InqQL}_2$ for $\mathsf{M}_{\mathrm{all}}$ to the completeness of $\mathbf{QLV}^+$ for $\mathsf{M}_{\mathrm{all}}$ as follows: Suppose $\mathsf{M}_{\mathrm{all}} \models A$. By the completeness of $\mathbf{QLV}^+$ for $\mathsf{M}_{\mathrm{all}}$, $\mathbf{QLV}^+ \vdash A$. Then, we can deduce by induction on the derivation for A that $\mathbf{T}\mathsf{InqQL}_2 \vdash\Rightarrow 0 : A$. This argument, however, does not give us a cut-elimination theorem of $\mathbf{T}\mathsf{InqQL}_2$.

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A A Proof of Lemma 9

Proof. The idea of this proof is essentially the same as in the proof of [4, Lemma 1]. The difference is: we need to care about $\forall$ and $\exists$. So, we basically concentrate on stating the difference from the proof of [4, Lemma 1] below. Suppose that a finite tree-sequent $\Gamma \Rightarrow \Delta$ is not provable in **cutfreeT**InqQL_2. In the following, we construct a sequence $(\Gamma^i \Rightarrow \Delta^i)_{i\in\omega}$ of finite tree-sequents and obtain $\Gamma^+ \Rightarrow \Delta^+$ as the union of them.

Let $(\alpha_i : F_i)_{i>1}$ be an enumeration of all labelled formulas such that each formula of $\mathcal{L}$ appears infinitely many times. We also enumerate all variables as $(x_i)_{i\in\omega}$ and all terms as $(t_i)_{i\in\omega}$. From now on, we construct $(\Gamma^i \Rightarrow \Delta^i)_{i\in\omega}$ such that **cutfreeT**$\mathsf{InqQL}_2 \nvdash \Gamma^i \Rightarrow \Delta^i$.
(Basis) Let $\Gamma^0 \Rightarrow \Delta^0 \equiv \Gamma \Rightarrow \Delta$. By assumption, **cutfreeT**$\mathsf{InqQL}_2 \nvdash \Gamma^0 \Rightarrow \Delta^0$.
(Inductive step) Suppose that we have already defined $\Gamma^{k-1} \Rightarrow \Delta^{k-1}$ such that **cutfreeT**$\mathsf{InqQL}_2 \nvdash \Gamma^{k-1} \Rightarrow \Delta^{k-1}$. In this k-th step, we define $\Gamma^k \Rightarrow \Delta^k$ so that unprovability of the tree-sequent is preserved. The operations executed in the k-th step are as follows: First, for any $0 : A \in \Gamma^k$, we add $1 : A$ and $2 : A$ to Γ^{k-1}. Unprovability is preserved because of the rule (Move). We denote the result of this step by $(\Gamma^{k-1})' \Rightarrow \Delta^{k-1}$. Second, according to the form of $\alpha_k : F_k$, one of the following operation is executed:

(1) The case where $F_k \equiv P(t_1, \dots, t_n)$ and $\alpha_k \neq 0$ and $\alpha_k : F_k \in (\Gamma^{k-1})'$. Define:

$$\Gamma^k \Rightarrow \Delta^k \equiv \begin{cases} 0 : P(t_1, \dots, t_n), (\Gamma^{k-1})' \Rightarrow \Delta^{k-1} & \text{if } (3 - \alpha_k) : P(t_1, \dots, t_n) \in (\Gamma^{k-1})'; \\ (\Gamma^{k-1})' \Rightarrow \Delta^{k-1} & \text{o.w.} \end{cases}$$

Unprovability is preserved because of (Atom L).
(2) The case where $F_k \equiv A \wedge B$ and $\alpha_k : F_k \in (\Gamma^{k-1})'$. See [4].
(3) The case where $F_k \equiv A \wedge B$ and $\alpha_k : F_k \in \Delta^{k-1}$. See [4].
(4) The case where $F_k \equiv A \vee B$ and $\alpha_k : F_k \in (\Gamma^{k-1})'$. Similar to (3).
(5) The case where $F_k \equiv A \vee B$ and $\alpha_k : F_k \in \Delta^{k-1}$. Similar to (2).
(6) The case where $F_k \equiv \neg A$ and $\alpha_k : F_k \in (\Gamma^{k-1})'$. See [4].
(7) The case where $F_k \equiv \neg A$ and $\alpha_k : F_k \in \Delta^{k-1}$. See [4].
(8) The case where $F_k \equiv A \rightarrow B$ and $\alpha_k : F_k \in (\Gamma^{k-1})'$. See [4].
(9) The case where $F_k \equiv A \rightarrow B$ and $\alpha_k : F_k \in \Delta^{k-1}$. See [4].
(10) The case where $F_k \equiv \forall x. A$ and $\alpha_k : F_k \in (\Gamma^{k-1})'$. Define $\Gamma^k \Rightarrow \Delta^k \equiv \alpha_k : A[t_0/x], \dots, \alpha_k : A[t_{k-1}/x], (\Gamma^{k-1})' \Rightarrow \Delta_k$. Unprovability is preserved because of ($\forall$L).
(11) The case where $F_k \equiv \forall x. A$ and $\alpha_k : F_k \in \Delta^{k-1}$. Take a fresh variable z, and define $\Gamma^k \Rightarrow \Delta^k \equiv (\Gamma^{k-1})' \Rightarrow \Delta_k, \alpha_k : A[z/x]$. Unprovability is preserved because of ($\forall$R).

(12) The case where $F_k \equiv \exists x. A$ and $\alpha_k : F_k \in (\Gamma^{k-1})'$. Similar to (11).
(13) The case where $F_k \equiv \exists x. A$ and $\alpha_k : F_k \in \Delta^{k-1}$. Similar to (10).
(14) Otherwise. It suffices to define $\Gamma^k \Rightarrow \Delta^k \equiv (\Gamma^{k-1})' \Rightarrow \Delta^{k-1}$.

Now let $\Gamma^+ \Rightarrow \Delta^+$ be $(\bigcup_{i\in\omega} \Gamma^i) \Rightarrow (\bigcup_{i\in\omega} \Delta^i)$. It is easy to verify that the tree-sequent $\Gamma^+ \Rightarrow \Delta^+$ is saturated. □

B A Proof of Lemma 10

By induction on the derivation of $\Gamma \Rightarrow \Delta$ in $\mathbf{T}\mathsf{InqQL}_2$. First, let us choose some fresh propositional variables **s**, **t** not occurring in the derivation. We assume that all formulaic translations in this proof depend on **s** and **t**. All cases in our induction immediately follow from the following Lemmas 12 and 13. We can easily establish Lemma 12 by definition of $[\![\Gamma \Rightarrow \Delta]\!]$.

Lemma 12. *If* $\mathsf{M}_{\mathrm{all}} \models [\![\Gamma \Rightarrow \Delta]\!]_\alpha$ *for some* $\alpha \in \{1, 2\}$, *then* $\mathsf{M}_{\mathrm{all}} \models [\![\Gamma \Rightarrow \Delta]\!]$.

Lemma 13. *The following formulas are valid in* $\mathsf{M}_{\mathrm{all}}$.

(**ax**) $A \wedge C \to A \vee D$.
(⊥**left**) $\bot \wedge C \to D$.
(**atom left**) $X_1 \to X_2$, *where:*
$X_1 \equiv P(t_1, \ldots, t_n) \to (S \vee T) \vee D \vee (S \wedge E \to T \vee F) \vee (T \wedge G \to S \vee H)$;
$X_2 \equiv (S \vee T) \vee D \vee (P(t_1, \ldots, t_n) \wedge S \wedge E \to T \vee F) \vee (P(t_1, \ldots, t_n) \wedge T \wedge G \to S \vee H)$.
(**move**) $((E \wedge A \to F) \vee (G \wedge A \to H)) \to (A \to (E \to F) \vee (G \to H))$.
(∧**right**) $(C \to D \vee A) \wedge (C \to D \vee B) \to (C \to (D \vee (A \wedge B)))$.
(∨**left**) $(A \wedge C \to D) \wedge (B \wedge C \to D) \to (((A \vee B) \wedge C) \to D)$.
(¬**left**) $(C \to D \vee A) \to (\neg A \wedge C \to D)$.
(¬**right**$_{1,2}$) $(C \wedge A \to D) \to (C \to D \vee \neg A)$.
(¬**right**$_0$) $X_3 \wedge X_4 \to X_5$, *where:*
$X_3 \equiv (S \vee T) \vee D \vee (S \wedge E \wedge A \to F \vee T) \vee (T \wedge G \to S \vee H)$;
$X_4 \equiv (S \vee T) \vee D \vee (S \wedge E \to F \vee T) \vee (T \wedge G \wedge A \to S \vee H)$;
$X_5 \equiv (S \vee T) \vee \neg A \vee D \vee (S \wedge E \to F \vee T) \vee (T \wedge G \to S \vee H)$.
(→ **left**) $(C \to D \vee A) \wedge (C \wedge B \to D) \to (C \wedge (A \to B) \to D)$.
(→ **right**$_{1,2}$) $(C \wedge A \to D \vee B) \to (C \to (D \vee (A \to B)))$.
(→ **right**$_0$) $(X_6 \wedge X_7 \wedge X_8) \to X_9$, *where:*
$X_6 \equiv A \to ((S \vee T) \vee D \vee B \vee (S \wedge E \to T \vee F) \vee (T \wedge G \to S \vee H))$;
$X_7 \equiv (S \vee T) \vee D \vee (S \wedge E \wedge A \to T \vee F) \vee (T \wedge G \to S \vee H)$;
$X_8 \equiv (S \vee T) \vee D \vee (S \wedge E \to T \vee F) \vee (T \wedge G \wedge A \to S \vee H)$;
$X_9 \equiv (S \vee T) \vee (A \to B) \vee D \vee (S \wedge E \to T \vee F) \vee (T \wedge G \to S \vee H)$.
(∀**left**) $(C \wedge A[t/x] \to D) \to (C \wedge \forall x. A \to D)$.
(∀**right**) $(C \to D \vee A[z/x]) \to (C \to D \vee \forall x. A)$, *where z is fresh in C, D and* $\forall x. A$.
(∃**left**) $(C \wedge A[z/x] \to D) \to (C \wedge \exists x. A \to D)$, *where z is fresh in C, D and* $\exists x. A$.
(∃**right**) $(C \to D \vee A[t/x]) \to (C \to D \vee \exists x. A)$.
(**cut**) $(C \to D \vee A) \wedge (C \wedge A \to D) \to (C \to D)$.

Proof. Formulas except (**atom left**), (¬ **right**$_0$) and (→ **right**$_0$) are all theorems of first-order intuitionistic logic with **CD** (we need **CD** for (∀ right)). Therefore, they are all valid in $\mathsf{M}_{\mathrm{all}}$. So, it suffices to check (**atom left**), (¬ **right**$_0$) and (→ **right**$_0$). The essential arguments for these are the same as in the propositional case [4, p.373, Lemma 3]. □