Logic Constants
Invariance for Modal and Dynamic Logic

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Invariance and constancy: 
*Logical Foundations for Interaction* 
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Invariance and constancy: Logical Foundations for Interaction

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The initial question:
- What is a logical constant?
  Why do we select $\land$, $\exists$, $\forall$ as logical expressions and build our logical systems around them?

The good old answer (Tarski’s):
- Because they have special semantic properties
  Quantifiers get interpreted by operations which are invariant under permutation.
The Gothenburg project (cont.)

Widen the horizon:

- consider other languages
- consider other objects
- consider alternative approaches to logicaity
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  What’s invariance for modal quantifiers?
  What’s the connection with invariance for FO quantifiers?
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- consider other languages
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  What are the natural ‘logical’ operations on games?
  What’s the connection with linear connectives?

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  What are the natural ‘logical’ operations on games?
  What’s the connection with linear connectives?

- consider alternative approaches to logicality
  From consequence relations to logical constants.
  Logicality as constancy
Some sample work:
invariance for modal and dynamic logic

- generalizes previous work on FO languages
- through a general perspective
  on invariance and logical systems

For today

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Languages and similarity relations

$L$ a logic, $S$ a ‘similarity relation’ for $L$
(equivalence relation on the class of $L$-structures)

1. $L$’s expressive power is bound by $S$,

2. In these limits, $L$ is as expressive as possible:

3. $S$-invariance as $L$’s logicality criterion.
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1. $L$’s expressive power is bound by $S$,
   - If $M \sim S M'$ then $M \equiv_L M'$

   $\implies$ Isomorphisms, Potential isomorphisms for FOL
   $\implies$ Bisimulations for Modal Logic

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2. In these limits, $L$ is as expressive as possible:
   - If $\mathcal{M} \equiv_L \mathcal{M}'$ then $\mathcal{M} \sim \mathcal{M}'$
   - $L$ is the strongest ‘finitary’ logic such that (1) holds.
   ⇒ Lindström Theorem
   ⇒ van Benthem characterization Theorem

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   $\implies$ Lindström Theorem
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3. $S$-invariance as $L$’s logicality criterion.
   - $S$-closed classes interpret logical operations.
   $\implies$ FO quantifiers and invariance under isomorphisms
A case in point

\[ L = FOL, \quad S = Iso_p \] (short for ‘being potentially isomorphic’)

**Definition**

\( f \) is a **partial isomorphism** between \( A \) and \( B \) just in case \( f \) is an isomorphism btw substructures of \( A \) and \( B \).
A case in point

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**Definition**

A **potential isomorphism** \( I \) between \( A \) and \( B \) is a nonempty set of partial isomorphisms s.t. for every \( f \in I \) and \( a \in A \) (resp. \( b \in B \)), there is \( g \in I \) with \( f \subseteq g \) and \( a \in \text{dom}(g) \) (resp. \( b \in \text{rng}(g) \)).
A case in point

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**Definition**

$f$ is a **partial isomorphism** between $A$ and $B$ just in case $f$ is an isomorphism btw substructures of $A$ and $B$.

**Definition**

A **potential isomorphism** $l$ between $A$ and $B$ is a nonempty set of partial isomorphisms s.t. for every $f \in l$ and $a \in A$ (resp. $b \in B$), there is $g \in l$ with $f \subseteq g$ and $a \in \text{dom}(g)$ (resp. $b \in \text{rng}(g)$).

Classical example: $\langle \mathbb{Q}, \leq \rangle$ and $\langle \mathbb{R}, \leq \rangle$ not isomorphic but potentially isomorphic.
A case in point (cont.)

- $L = \text{FOL}$, $S = \text{Iso}_p$

An obvious but elusive parallel:

- $L = \text{atoms, booleans} + \exists$
- $S = \text{partial isomorphisms} + \text{picking one more}$
A case in point (cont.)

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An obvious but elusive parallel:
- $L = \text{atoms, booleans}$ + $\exists$
- $S = \text{partial isomorphisms}$ + picking one more

- $L = \text{ML}$, $S = \text{BiS}$ (short for ‘being bisimilar’)

Same thing:
- $L = \text{atoms, booleans}$ + $\Diamond$
- $S = \text{world matching}$ + moving along $R$
Atom preservation

\[
\langle M, P, a \rangle \xrightarrow{S} \langle M', P', a' \rangle \\
\text{iff} \quad a \in P \iff a' \in P'
\]
Atom preservation

\[
\langle M, P, a \rangle \xrightarrow{S} \langle M', P', a' \rangle
\]

\[a \in P \text{ iff } a' \in P'\]

**Definition**

A similarity relation \( S \) **preserves atoms** iff for all \( S \)-similar structures, distinguished relations behave similarly on distinguished objects.
Commutation with object expansions

\[ M, a S M', a' \]
Commutation with object expansions

Definition
A similarity relation $S$ commutes with object expansions iff if $M \xrightarrow{S} M'$, then for all $a \in |M|$, there is an $a' \in |M'|$ s.t. $M, a S M', a'$. 
Ordering on similarity relations

Definition

\[ S \leq S' \iff S' \subseteq S. \]
Ordering on similarity relations

Definition

\[ S \leq S' \text{ iff } S' \subseteq S. \]

Ex: Universal relation \( \leq \text{Iso}_p \leq \text{Iso} \).
Characterization of $Iso_p$

**Fact**

$ Iso_p$ is the smallest similarity relation $S$ such that

- $S$ preserves atoms
- $S$ commutes with objects expansions.
Back to the logic

How does this connect with properties of first-order languages?
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Atoms preservation says that similar structures are elementary equivalent on atomic sentences and boolean compounds thereof.
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What does commutation says?
How does this connect with properties of first-order languages?

Atoms preservation says that similar structures are elementary equivalent on atomic sentences and boolean compounds thereof.

What does commutation says?

It says that existential quantification is in the language.
Object projection

Definition

Let $Q$ be a class of structures of the form $M, a$. The **object projection** of $Q$, $\exists(Q)$, is defined by $M \in \exists(Q)$ iff there is a $b \in |M|$ such that $M, b \in Q$.

This is what you can do with $\exists$. 
Object projection

Definition

Let $Q$ be a class of structures of the form $M, a$. The object projection of $Q$, $\exists(Q)$, is defined by $M \in \exists(Q)$ iff there is a $b \in |M|$ such that $M, b \in Q$.

This is what you can do with $\exists$.

$\exists$ is logical means $\exists$ does not break invariance:

Definition

Object projection preserves $S$-invariance iff whenever $Q$ is $S$-invariant, so is $\exists(Q)$. 
Equivalence result

Theorem

\[ S \text{ commutes with object expansions} \]
\[ \iff \]
\[ \text{object projection preserves } S\text{-invariance}. \]
The Gothenburg project
Invariance for FO operations
Higher up and back
Dynamic Logic
Conclusion

Equivalence result

Theorem

\[ S \text{ commutes with object expansions} \iff \text{object projection preserves } S\text{-invariance.} \]

Corollary

\( Iso_p \) is the smallest similarity relation \( S \) such that

- \( S \) preserves atoms
- object projection preserves \( S\)-invariance.
Theorem

\[ S \text{ commutes with object expansions} \]
\[ \text{iff} \]
\[ \text{object projection preserves } S\text{-invariance.} \]

Corollary

Iso\(_p\) is the smallest similarity relation \( S \) such that

- \( S \) preserves atoms
- object projection preserves \( S\)-invariance.

\[ \Rightarrow \text{Iso}\(_p\) is the good match for a language based on } \exists. \]
A general setting

- A a class of objects
- E a relation on A
- S an equivalence relation on A
- $E^{-1} : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ an inverse for E
  defined for $X \subseteq A$ by $E^{-1} = \{ a \in A / \exists b \in X \text{ with } aEb \}$

Definition

A subclass $X$ is **S-invariant** iff if $a \in X$ and $aSb$ then $b \in X$. 
Commutation

**Definition**

*S commutes with* $E$ iff, for all $a, a', b \in A$, if $aSb$ and $aEa'$, then there is a $b'$ such that $a'Sb'$ and $bEb'$.
Preservation of Invariance

**Definition**

\( E^{-1} \) preserves **S-invariance** iff for any subclass \( X \) of \( A \), if \( X \) is \( S \)-invariant, then \( E^{-1}(X) \) is \( S \)-invariant.

\[
\begin{align*}
X & \quad \text{S-invariant} \\
E^{-1}(X) & \quad \text{S-invariant}
\end{align*}
\]
Commutation lemma

**Lemma**

\( S \text{ commutes with } E \text{ iff } E^{-1} \text{ preserves } S\text{-invariance.} \)
Commutation lemma

Proof

$S$ commutes with $E$, $E^{-1}$ preserves $S$-invariance

$\Rightarrow$

$$a \in E^{-1}(X) \xrightarrow{S} b$$
Commutation lemma

Proof

\( S \) commutes with \( E \), \( \exists E^{-1} \) preserves \( S \)-invariance \( \exists \)

\[ a' \in X \]
\[ a \in E^{-1}(X) \xrightarrow{S} b \]
Commutation lemma

Proof

$S$ commutes with $E$, $E^{-1}$ preserves $S$-invariance

$\Rightarrow$

\[
\begin{align*}
a' &\in X \\ b' &\in X
\end{align*}
\]

\[
\begin{align*}
a &\in E^{-1}(X) \\ b &\in X
\end{align*}
\]
Commutation lemma

Proof

$S$ commutes with $E$, $E^{-1}$ preserves $S$-invariance.

$\Rightarrow$

\[
\begin{align*}
  a' &\in X & S &\quad b' &\in X \\
  a &\in E^{-1}(X) & S & b
\end{align*}
\]
Commutation lemma

Proof

\( S \) commutes with \( E \), ? \( E^{-1} \) preserves \( S \)-invariance ?

\[
\begin{align*}
\Rightarrow \\
\begin{array}{c}
a' \in X \\
\end{array} \\
\begin{array}{c}
a \in E^{-1}(X) \\
\end{array}
\end{align*}
\]

\[
\begin{array}{c}
S \quad \quad \quad b' \in X \\
\end{array} \\
\begin{array}{c}
b \in E^{-1}(X) \\
\end{array}
\]

\[
\begin{array}{c}
E \\
\end{array} \\
\begin{array}{c}
E^{-1} \\
\end{array}
\]
Commutation lemma

**Proof**

\( S \) commutes with \( E \), \( E^{-1} \) preserves \( S \)-invariance
Commutation lemma

Proof

? S commutes with E ?, $E^{-1}$ preserves S-invariance

\[ a' \in [a']_S \]

\[ \begin{array}{c}
E \\
[\text{Diagram}]
\end{array} \]

\[ a \quad S \quad b \]
Commutation lemma

Proof

\( S \) commutes with \( E \), \( E^{-1} \) preserves \( S \)-invariance

\[ a' \in [a']_S \]

\[ a \in E^{-1}([a']_S) \xrightarrow{S} b \]
Commutation lemma

Proof

$S$ commutes with $E$, $E^{-1}$ preserves $S$-invariance

\[ a' \in [a']_S \]
\[ E \]
\[ a \in E^{-1}([a']_S) \xrightarrow{S} b \in E^{-1}([a']_S) \]
Commutation lemma

Proof

? S commutes with E ?, $E^{-1}$ preserves S-invariance

$\Leftarrow$

$\begin{align*}
\forall a' \in [a']_S \quad \forall b' \in [a']_S \\
E^{-1}( [a']_S ) &\xrightarrow{S} b \in E^{-1}( [a']_S )
\end{align*}$
Commutation lemma

Proof

$S$ commutes with $E$, $E^{-1}$ preserves $S$-invariance

$\leftarrow$

$a' \in [a']_S \overset{S}{\longrightarrow} b' \in [a']_S$

$E \uparrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 
Using the lemma

Take

- $A$ the class of FO structures
- $E$ expanding with one object
  $\Rightarrow \ E^{-1}$ is object projection.

As an instance of the commutation lemma, we get:

**Theorem**

$S$ commutes with object expansions

iff

object projection preserves $S$-invariance.
The modal case

Definition

A similarity relation $S$ commutes with guarded object expansion iff, if $\mathcal{M}, w \ S \mathcal{M}', w'$, then for all $v \in |\mathcal{M}|$ with $w R v$, there is a $v' \in |\mathcal{M}'|$ such that $\mathcal{M}, v \ S \mathcal{M}', v'$ and $w' R' v'$. 
Characterization of bisimulations

*BiS* short for ‘being bisimilar’

**Fact**

*BiS is the smallest similarity relation S such that*

- $S$ preserves atoms
- $S$ commutes with guarded object expansion.
Guaranteed object projection

Definition

Let \( Q \) be a class of pointed Kripke structures. The **guaranteed object projection** of \( Q \), \( \diamond(Q) \), is defined by

\[
M, w \in \exists(Q) \text{ iff there is a } v \in |M| \text{ such that } M, v \in Q \text{ and } wRv.
\]

This is what you can do with \( \diamond \).
Guar ded object projection

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\]

This is what you can do with \( \Diamond \).

\( \Diamond \) is logical *means* \( \Diamond \) does not break invariance:

Definition

Guarded object projection **preserves** \( S \)-invariance iff whenever \( Q \) is \( S \)-invariant, so is \( \Diamond(Q) \).
Equivalence result

- A the class of pointed Kripke structures
- $E$ moving to an accessible world
  $\Rightarrow E^{-1}$ is guarded object projection.

As an instance of the commutation lemma, we get:

**Theorem**

*S commutes with guarded object expansion*  
iff  
*guarded object projection preserves S-invariance.*
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As an instance of the commutation lemma, we get:

**Theorem**

$S$ commutes with guarded object expansion
iff

*guarded object projection preserves $S$-invariance.*

**Corollary**

* Bis is the smallest similarity relation $S$ such that
  - $S$ preserves atoms
  - *guarded object projection preserves $S$-invariance.*
Equivalence result

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- guarded object projection preserves $S$-invariance.
Dynamic logic

The language of Propositional Dynamic Logic (PDL)

**Programs**

\[ \pi \ ::= \ R \mid \pi ; \pi \mid \pi \cup \pi \mid \pi^* \mid ?\phi \]

**Formulas**

\[ \phi \ ::= \ p \mid \neg\phi \mid \phi \land \phi \mid \langle \pi \rangle \phi \]
Dynamic logic

The language of Propositional Dynamic Logic (PDL)

**Programs**

\[ \pi ::= R \mid \pi; \pi \mid \pi \cup \pi \mid \pi^* \mid ?\phi \]

**Formulas**

\[ \phi ::= p \mid \neg\phi \mid \phi \land \phi \mid \langle\pi\rangle\phi \]

- ML formulas can only define set of worlds
- PDL programs can also define relations
What is a PDL operation?

Look at what $\cup$ does on a fixed set of worlds $W$:

$$\| \cup \|_W : \wp(W^2) \times \wp(W^2) \rightarrow \wp(W^2)$$

$$R \times R' \leftrightarrow R \cup R$$
What is a PDL operation?

Look at what $\cup$ does on a fixed set of worlds $W$:

$$|| \cup ||_W : \wp(W^2) \times \wp(W^2) \rightarrow \wp(W^2)$$

$$R \times R' \mapsto R \cup R$$

In general: a dynamic operator $\overline{O}$ is interpreted by a function $O$ from Kripke models to relations over these models.
What is a PDL operation?

Look at what $\cup$ does on a fixed set of worlds $W$:

$$\mathcal{P}(W^2) \times \mathcal{P}(W^2) \rightarrow \mathcal{P}(W^2)$$

$$R \times R' \mapsto R \cup R$$

In general: a dynamic operator $\overline{O}$ is interpreted by a function $O$ from Kripke models to relations over these models.

So let $\overrightarrow{\chi}$ be a sequence of programs and formulas matching the syntactic type of $\overline{O}$,

The semantic clause for $\overline{O}$ is given by:

$$\overline{\mathcal{O}\overrightarrow{\chi}}_M = O(\overline{M}, \overline{\overrightarrow{\chi}}_M)$$
Safety

For any bisimulation $Z$:

\[ M, v \xrightarrow{Z} M', v' \]

\[ O(M) \]

\[ M, w \xrightarrow{Z} M', w' \]

\[ O(M') \]

**Definition**

A dynamic operation $O$ is safe for bisimulation iff whenever $Z$ is a bisimulation between $M, w$ and $M', w'$, and $wO(M) \nu$ for some $\nu \in |M|$, then there is a $\nu' \in |M'|$ such that $\nu Z \nu'$ and $w' O(M') \nu'$. 
Safety and PDL

- Safety is the key element in the proof that PDL formulas are invariant under bisimulation.
- Enriching PDL programs with new safe operations yields extensions which are still invariant under bisimulation.
- PDL without Kleene star is the safe fragment of FOL.
Safety and PDL

- Safety is the key element in the proof that PDL formulas are invariant under bisimulation.

- Enriching PDL programs with new safe operations yields extensions which are still invariant under bisimulation.

- PDL without Kleene star is the safe fragment of FOL.

However, safety is not our standard commutation property.
Commutation with $\text{Bis}$

\[
\begin{array}{c}
\mathcal{M}, v \xrightarrow{\text{BiS}} \mathcal{M}', v' \\
\mathcal{M}, w \xrightarrow{\text{BiS}} \mathcal{M}', w'
\end{array}
\]

Definition

A dynamic operation $O$ commutes with $\text{BiS}$ iff whenever $\mathcal{M} + \|\vec{x}\|_\mathcal{M}$, $w$ and $\mathcal{M}' + \|\vec{x}\|_\mathcal{M}'$, $w'$, and $wO(\|\vec{x}\|_\mathcal{M})v$ for some $v \in |\mathcal{M}|$, then there is a $v' \in |\mathcal{M}'|$ such that $vZv'$ and $w'O(\|\vec{x}\|_\mathcal{M}')v'$. 
As before, we can get:

**Theorem**

\( \text{BiS commutes with } O \text{ iff } O \text{ preserves invariance under BiS.} \)
The lemma again

As before, we can get:

**Theorem**

\( \text{BiS commutes with } O \iff O \text{ preserves invariance under BiS.} \)

Preserving invariance under \( \text{BiS} \) is precisely what we need if we want to stay within the realm of modal logic.
Safety and commutation

How does this relate to safety?
Safety and commutation

How does this relate to safety?

**Theorem**

\textit{BiS commutes with }O\textit{ iff }O\textit{ is safe for bisimulation.}

Safety is indeed the natural constraint on dynamic operations \textit{qua} modal.
Conclusion

- *a general perspective on invariance*,
  ‘commutation lemma’
  easy but nice result for $Iso_p$ and $BiS$
  nice and not so easy result for safety

- *stemming from some sort or ‘reverse’ meta-logic*,
  Duality btw syntax and semantics
  Take $S$ as a parameter

- *to be developed*...
  Apply to other logical systems
  Generalize to games via game logic
  Connect to ‘intrinsic’ regularities