Should we fear Benacerraf's multiple reducibility challenge? Definitions and logical forms in philosophy of mathematics.

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Benacerraf's Dilemma

Benacerraf 1973:

Two quite distinct kinds of concerns motivated accounts of the nature of mathematical truth :

"(1) the concern for having a homogeneous semantical theory in which semantics for the propositions of mathematics parallel the semantics for the rest of the language, and

(2) the concern that the account of mathematical truth mesh with a reasonable epistemology."

Benacerraf argued that one of these masters can only be served at the expense of the other.

An homogeneous semantical theory

Benacerraf 1973:

"Consider the following two sentences:

(1) There are at least three large cities older than New York

(2) There are at least three perfect numbers greater than 17.

Do they have the same logicogrammatical form? More specifically, are they both of the form

(3) There are at least three FG's that bear R to a,

(…) What are the truth conditions of (1) and (2)? Are they relevantly parallel?"

"I suggest that, if we are to meet (the requirement of having an homogeneous semantic), we shouldn't be satisfied with an account that fails to treat (1) and (2) in parallel fashion, on the model of (3).

There may be *differences*, but I expect these to emerge at the level of the analysis of the reference of the singular terms and predicates."

A weak reading of the semantic syntaxic constraint

Hale & Wright, *European Journal of Philosophy*, 10:1, 2002, pp. 101-129. 102:

"Taken on its own, this suggests a very exacting reading of the semantic constraint, under which it can be satisfied only by an account of the truth-conditions of (the math statement) which respects its surface syntax exactly as just described.

It is doubtful that Benacerraf can have intended quite such a demanding interpretation. Certainly, so understood, it would go well beyond the somewhat vague demand that 'whatever semantical account we are inclined to give of … singular terms, predicates, and quantifiers in the mother tongue include those parts of [it] we classify as mathematese' [408]. That would seem to come to no more than the weaker requirement—which Benacerraf certainly does endorse—that an account of the truthconditions of mathematical statements should accord **with a broadly referential (i.e. Tarskian) semantics for the language as a whole**.

This would leave room for accounts of mathematics which view the surface grammatical form of mathematical statements as **more or less misleading** as to their logical form, provided that their (alleged) logical form involves only devices amenable to Tarskian treatment."

A strong reading of the semantic syntaxic constraint

Shapiro 2006:

A) “Before going further, I would like to acknowledge an orientation. As I see it, the goal of philosophy of mathematics is to *interpret* mathematics, and articulate its place in the overall intellectual enterprise. One desideratum is to have an interpretation that takes as much as possible of what mathematicians say about their subject as **literally true, understood at or near face value**. **Call this the *faithfulness* constraint.**”

B) “A second, and weaker, desideratum is to develop an interpretation that does not go too much beyond what mathematicians say about their subject. (…) Presumably, philosophical questions about mathematics are not to be answered solely in mathematical terms. The second desideratum is to **not attribute *mathematical* properties to mathematical objects unless those attributions are explicit or at least implicit in mathematics itself. Call this the *minimalism* constraint.**”

A link with the multiple reductions challenge

Shapiro 2006:

Richard Dedekind's philosophical methodology was to develop a system of objects, and then abstract the structure of the system (e.g., Dedekind [1872], [1888]). For example, he constructed the system of cuts in rationals, and then abstracted the real numbers from the cuts. According to Dedekind, the abstracted items— the real numbers— are not part of the system abstracted from, but are instead something "new" which the mind freely "creates" (see Shapiro [1997, Chapter 5, §4]). Dedekind's friend Heinrich Weber suggested instead that real numbers be *identified* with cuts. Dedekind replied that there are many properties that cuts have which would sound very odd if applied to the corresponding real numbers (Dedekind [1932, Vol. 3, 489-490]). For example, cuts have members. Do real numbers have members? Dedekind's **Benacerraf-type point** makes sense in the context of the minimalism constraint.

Benacerraf 1965:

In ZF, 2 is defined as $\{\{\emptyset\}\}$, while, in von Neumann's approach, 2 is defined as $\{\emptyset, \{\emptyset\}\}$. Which is the genuine 2? Has 2 one or two elements? This is not a question mathematicians usually ask, and therefore a good philosophy of math should exclude the question from its agenda.

The strong reading and the respect of the mathematical practices

- Benacerraf and Shapiro develop a strong reading of the semantic syntaxic constraint, according to which the demand of taking the math statement as face value is presented as a direct consequence of the requirement that **philosophers of math should respect what mathematicians say** about their concepts (about natural numbers, about real numbers, etc.).

- More precisely, both Benacerraf and Shapiro select, for the concept they consider, a standard context of use (the ω-sequence for the natural numbers, the complete ordered field for the real numbers), and sustain that, this context being granted, the philosophical accounts which have "less excess baggage than the others" (which do not introduce foreign considerations) are the best ones.

All this is related to the multiple reductions challenge.

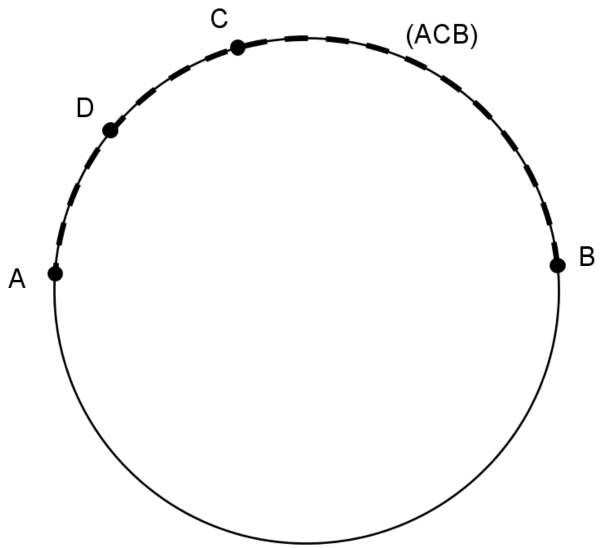
Some doubts

- What mathematicians say about their concepts is rarely univocal. Mathematicians use often different languages to speak about their concepts. In order to use their constraint, Benacerraf and Shapiro have then to make a choice between the various ways mathematicians express themselves.

- Now, this choice of a language is often related to the choice of a standard context in which a given concept should be embedded. Now this question of delineating the a proper context for a given concept is itself a math question, which cannot be considered as already solved by the philosopher.

In other words, the problem of multiple reductions challenge is not a question which arises for the philosopher when philosopher does bad metaphysics and forgets the math material – it is a problem mathematicians themselves encounter, as soon as they face "architectonic" issues.

An example: Pieri’s definition of order on projective line



D belongs to the segment (ACB)

An example: Pieri's definition of order on projective line

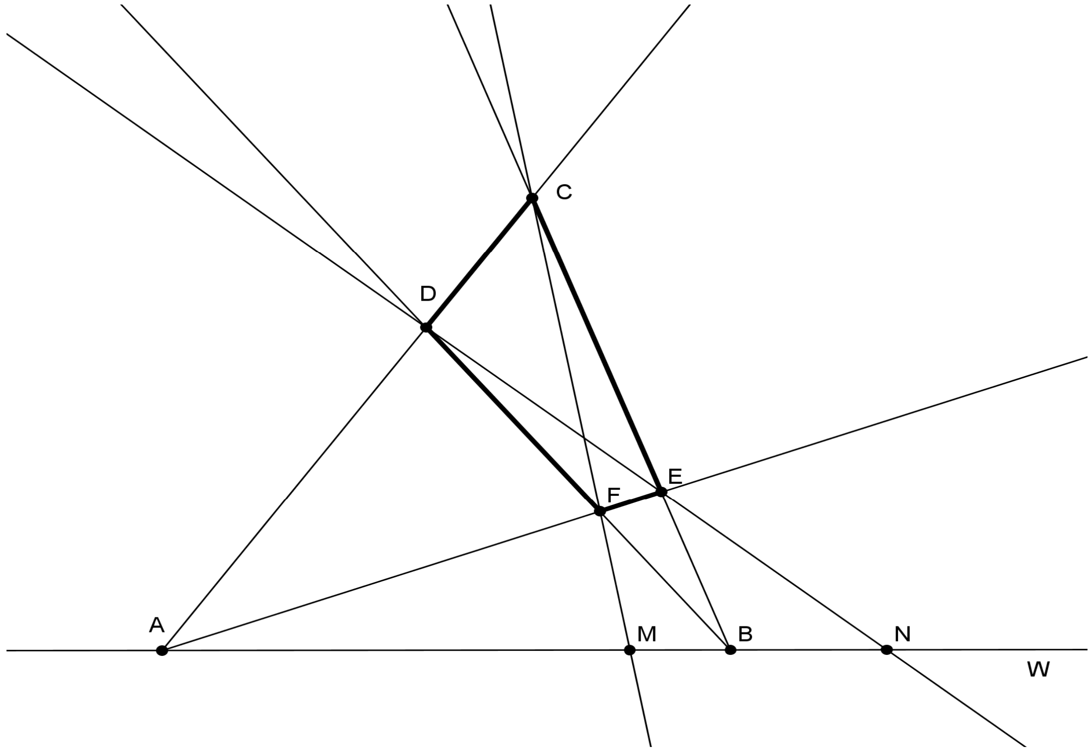
D belongs to the segment (ABC) iff there are two points M, N on the line (AB) such that A, B and C, D both are harmonic conjugates with respect to M and N.

$$D \in (ABC) =_{Def} \exists M \exists N \in (AB),\ A H_{M,N} B \ \&\ C H_{M,N} D.$$

The form on the left is not the same as the form on the right.

An example: Pieri's definition of order on projective line

A, B are harmonic conjuguate in respect with M, N, if there is a quadrilateral (CDFE) such that the two couples of opposing sides intersect at A and B and such that the two diagonals CF and DE intersect the line (AB) in M and N.



Pieri's achievement

To summarize: in the standard presentation, projective geometry arises from a combination of two different axiom groups, an incidence and an ordinal one. In Pieri's approach, projective geometry is derived from only one indefinable relation, the incidence relation. This is truly beautiful: **the projective order on a line is derived from the way the lines intersect in the plane**.

A very important mathematical advance:

1) the fundamental theorem of projective geometry can be derived without introducing ordinal considerations – contrary to what pretended F. Klein.
2) the idea of deriving order from the distinction between square and not-square elements anticipated the construction of the purely algebraic theory of the real field by Artin and Schreier (Sinaceur 1991)

Back to the semantic syntaxic constraint

1) According to the semantic syntaxic constraint, projective geometry (in its standard formulation) speaks about order. But according to Pieri, projective geometry does not speak about order. Which is the right conception?

   My point is that **you cannot dismiss** this question as a purely metaphysical one.

   I agree that the example of multiple reductions taken by Benacerraf in his paper is not an interesting one. But it is uninteresting, not because the issue is purely metaphysical – but because the example is badly chosen.

Back to the semantic syntaxic constraint

2) Pieri's aim is to resume the work of Von Staudt, that is, to purify projective geometry from any foreign notions (for instance, metrical ones). There are thus some architectonic considerations behind Pieri's work: he wanted to guarantee the fact that projective geometry could be seen as a wholly independent theory.

Taking the statement "D belongs to (ACB)" at face value is OK when projective geometry is considered as the context in which the sentence takes place; but when the issue is to question the place of projective geometry within the building of geometry as a whole, then the analysis of order in term of incidence relation is, according to Pieri, better.

A more positive suggestion

In order to reduce a given mathematical theory T to another theory T' (for instance, in Pieri's reduction, the 'underlying' theory T' is incidence geometry; in logicism, logic is the 'underlying' logic), one has to define the underlying theory T'. But this is not enough. One has to determine as well which constraints one wants to put on the **paraphrase** function, which translate sentences of T in the language $L_{T'}$ of the underlying theory.

Rayo 2005: a paraphrase function * is **minimally** adequate for a math language L_T with an intended model M

(i) if the restriction of * to L_T is recursive (recursivity of the translation).

(ii) if there is a model S such that, for any sentence ϕ of L_T, $\vDash_M \phi$ if and only if $\vDash_S \phi$ (preservation of truth).

I suggest to viewing Benacerraf's and Shapiro's semantic syntaxic constaint as a **maximal** condition put on the adequacy of the paraphrase function. For the two philosophers, an acceptable paraphrase would be one which does not change the syntaxic form of the sentences of the theory T. Only identity function would then satisfy this maximally adequate requirement.

My point here would be to suggest that, between these two extreme constraints, many other conditions could be defined. To bring out and to explore these numerous intermediate cases could lead to interesting philosophical insight.

A reactivation of certain traditions

1) In Russell, where the math statements are not taken at face value, some stronger conditions are put on the paraphrase function – as for instance the Application Constraint (the paraphrase should explain the main applications of the theory). See Principia (1910-1913).

2) In the work of the Polish logician Lindenbaum, various criteria of simplicity are set forth and explored (reducing the number of indefinables, reducing the logical type of the notions paraphrased, reducing the number of terms of the relational indefinables, etc.). See Sur la simplicité formelle des notions (1935).

Conclusion: a too strong reading of the semantic syntaxic constraint could be a conceptual barrier which forbids using logic to account for certain features of math practices.

Lindenbaum 1936, p. 32:

According to a notorious metaphor from Vailati, nowadays, a democratic system prevails in the deductive theories: no terms own the right to rise above the others as the primitive term from which all the others ones depend. And so it is because the choice of the terms we acknowledge as primitives relating to a given theory is somewhat arbitrary (…).
[**This is akin to Benacerraf's multiple reductions challenge**]

I would say, however, that such a mechanical democratic standpoint has not aged well. The time has come for the terms which are best qualified for governing to take the power. One of the criteria (but not the only one) allowing us to choose among the terms is their intrinsic simplicities – or rather, the intrinsic simplicity of the system of terms, since usually it is a few independent terms which together do their job.
[**Taking up the challenge opens some new possibilities to use logic in philo of math**]

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