Some Funny Complexity Results for Judgment Aggregation

Wojtek Jamroga, University of Luxembourg
(joint work with Marija Slavkovik)

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Outline

1. Introduction
2. Distance-Based Judgment Aggregation
3. Complexity: Bad News
4. Positive Results
5. Conclusions

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1 Introduction

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Introduction

- It is often convenient to ascribe mental attitudes to groups of agents
- Examples: opinion of the government, belief of a religious group, goal of a company

collective of agents $\longrightarrow$ collective agent

Judgment Aggregation

- Similar to preference aggregation & voting
- Two approaches to judgment aggregation:
- Idealistic: specify postulates and prove impossibility
- Pragmatic: use a reasonably good procedure
- In the latter case, complexity is important!

Distance-Based Judgment Aggregation

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- Aggregation rules must be “well behaved” mathematically
- Does that imply that they are well-behaved computationally?

Distance-Based Judgment Aggregation

- Distance-based judgment aggregation defines the collective opinion as a well-behaved compromise between individual opinions
- Aggregation rules must be “well behaved” mathematically
- Does that imply that they are well-behaved computationally?
- Not necessarily...

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Judgment Aggregation

Definition (Judgment aggregation)

Let N be a finite set of agents, $\mathcal{A} \subseteq \mathcal{L}$ a finite agenda of issues from a propositional language $\mathcal{L}$, $\mathcal{C} \subseteq \mathcal{L}$ a finite set of admissibility constraints, and T a set of truth values.

Judgment sets (JS) are consistent and admissible combinations of opinions on issues from $\mathcal{A}$, that is, all $js : \mathcal{A} \to T$ such that there is a valuation $v \in PV$ with: (i) $val_v(\varphi) = js(\varphi)$ for every $\varphi \in \mathcal{A}$, and (ii) $val_v(\psi) = 1$ for every $\psi \in \mathcal{C}$.

A judgment profile is a collection of $|N|$ judgment sets, one per agent.

A judgment aggregation rule $\nabla : JS^{|N|} \to \mathcal{P}(JS) \setminus \{\emptyset\}$ aggregates opinions from all the agents into a collective judgment set (or sets).

Example: Guarding Robots

3 robots are guarding a building, and have just observed a person. Each robot must assess whether the person is authorized to be there (proposition $auth$), if it has malicious intent (mal), and whether to classify the event as dangerous intrusion ($intr$). Additionally, it is assumed that a non-authorized person with malicious intent implies intrusion: $\neg auth \wedge mal \rightarrow intr$.

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	$auth$	mal	$intr$
robot 1	1	1	0
robot 2	0	0	0
robot 3	0	1	1
majority	0	1	0

Note that the most obvious aggregation rule (majority) results in an inadmissible judgment set.

Distance-Based Aggregation

Definition (Distance-based judgment aggregation)

A distance-based aggregation rule looks for a collective opinion that does not stray too much from the individual judgments:

$$\nabla_{d,aggr}(jp) = \text{argmin}_{js \in JS} \left\{ aggr\big(d(js, jp[1]), \ldots, d(js, jp[|N|])\big) \right\},$$

where d is a distance metric, and $aggr$ an aggregation function.

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Take $aggr = \sum, \quad d = d_H$

Distance-Based Aggregation

Definition (Distance metric)

A distance over X is a function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ such that:
(minimality) $d(x, y) = 0$ iff $x = y$,
(symmetry) $d(x, y) = d(y, x)$, and
(triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$.

Two well known distances over $\{0, 1\}^m$ are: the Hamming distance d_H, and the drastic distance d_D

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Definition (Aggregation function)

An aggregation is a function $aggr : (\mathbb{R}^+)^n \to \mathbb{R}^+$ such that:
(minimality) $aggr(0^n) = 0$, and
(monotonicity) if $x \leq y$, then $aggr(\ldots, x, \ldots) \leq aggr(\ldots, y, \ldots)$.

Well known aggregators are: min, max, sum, and product.

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Complexity of winner set verification

Definition (winner set verification)

WINVER$_\nabla$ is the decision problem defined as follows:

Input: Agents N, agenda $\mathcal{A}$, constraints $\mathcal{C}$, judgment profile $jp \in JS^{|N|}(\mathcal{A},\mathcal{C})$, and judgment set $js \in JS(\mathcal{A},\mathcal{C})$;

Output: *true* if $js \in \nabla(jp)$, else *false*.

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What is the complexity of WINVER?

Bad News

Theorem

There is a distance which is not Turing computable.

Bad News

Proof. We construct the Turing distance d_{TR} as follows. First, we assume a standard encoding of Turing machines in binary strings; we use $TM(X)$ to refer to the machine represented by the string of bits $X \in \{0,1\}^m$. We also assume by convention that strings starting with 0 or ending with 1 represent only machines that always halt (e.g., some TM's with only accepting states).

Let $halts(X) = 0$ if the $TM(X)$ halts, and 1 otherwise. Now, for any $js, js' \in \{0,1\}^m$, we take

$$d_{TR}(js, js') = d_D(js, js') + halts(h(js, js')),$$

where d_D is the drastic distance, and $h(js, js')$ is the Hamming sequence for (js, js').

Bad News

Proof ctd. We check that d_{TR} is a distance metric:

1. $d_{TR}(js, js) = d_D(js, js) + halts(0^m) = 0$;
2. $d_{TR}(js, js') = 0 \Rightarrow d_D(js, js') = 0 \Rightarrow js = js'$;
3. $d_{TR}(js, js') = d_{TR}(js', js)$: straightforward;
4. Triangle inequality: the nontrivial case is $js \neq js' \neq js''$, then $d_{TR}(js, js') + d_{TR}(js', js'') \geq 2 \geq d_{TR}(js, js'')$.

For incomputability, we observe that $TM(X)$ halts iff $d_{TR}(X, 0^{|X|}) \leq 1$.

Bad News

Theorem

There is a distance and an aggregation function for which WINVER *is undecidable.*

Bad News

Proof. We construct a reduction from the halting problem. Given is a representation $X \in \{0,1\}^m$ of a Turing machine (same assumptions on the encoding). We take $d = d_{TR}$, $aggr = \sum$.

Let $\mathcal{A} = \{p_1, \ldots, p_m\}$ consist of n unrelated atomic propositions, $\mathcal{C} = \emptyset$, and $jp = \{0^m, X\}$. Now, for $X = 1 \ldots 0$ (the other cases of X trivially halt), we have that $TM(X)$ halts iff $js = 0^m, X$ are the only winners. This is because the aggregate scores of 0^m and X are 1 if $TM(X)$ halts and 2 otherwise, and no score can be less than 1. Moreover, for all other candidates $Y \in \{0,1\}^m$ the score is at least 2, and in particular for $Y = (1)^m$ it is always 2.

Suppose now that deciding WINVER terminates in finite time. Then, the halting of $TM(X)$ could be verified by 2^m WINVER checks, i.e., also in finite time – which is a contradiction.

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Positive Results

Theorem

If $aggr$ and d are computable in polynomial time then WINVER *for $\nabla_{d,aggr}$ is in $\mathbf{P}^{\mathbf{NP}[2]}$.*

Positive Results

Theorem

If $aggr$ and d are computable in polynomial time then WINVER *for $\nabla_{d,aggr}$ is in $\mathbf{P}^{\mathbf{NP}[2]}$.*

$\mathbf{P}^{\mathbf{NP}[k]}$ is the class of problems solvable by a polynomial-time deterministic Turing machine asking at most k adaptive queries to an $\mathbf{NP}$ oracle

For complexity freaks: $\mathbf{NP} \subseteq \mathbf{P}^{\mathbf{NP}[k]} \subseteq \mathbf{\Delta_2^P} = \mathbf{P}^{\mathbf{NP}}$

Algorithm: $\underline{Winver(js, jp, N, \mathcal{A}, \mathcal{C}, d, aggr)}$

1. if $Consistent(js, \mathcal{A}, \mathcal{C})$ and not $ExistsBetter(js, jp, N, \mathcal{A}, \mathcal{C}, d, aggr)$ then $return(true)$ else $return(false)$;

Oracle: $\underline{Consistent(js, \mathcal{A}, \mathcal{C})}$

1. guess a valuation $v \in PV$ for the atomic propositions in $\mathcal{A}$;
2. if $val_v(\varphi) = js(\varphi)$ for every $\varphi \in \mathcal{A}$ and $val_v(\psi) = 1$ for every $\psi \in \mathcal{C}$ then $return(true)$ else $return(false)$;

Oracle: $\underline{ExistsBetter(js, jp, N, \mathcal{A}, \mathcal{C}, d, aggr)}$

1. guess $js' \in JS$;
2. guess a valuation $v' \in PV$ for the atomic propositions in $\mathcal{A}$;
3. if $val_{v'}(\varphi) = js'(\varphi)$ for every $\varphi \in \mathcal{A}$ and $val_{v'}(\psi) = 1$ for every $\psi \in \mathcal{C}$ and $aggr\big(d(js', jp[1]), \ldots, d(js', jp[|N|])\big) < aggr\big(d(js, jp[1]), \ldots, d(js, jp[|N|])\big)$ then $return(true)$ else $return(false)$;

Algorithm: $\underline{Winver(js, jp, N, \mathcal{A}, \mathcal{C}, d, aggr)}$

Idea: ask the oracle if js is consistent and admissible
and whether there is no set with a better score

Good News

For typical distances and aggregation functions, we get the following as a straightforward consequence:

Corollary

If $aggr \in \{\min, \max, \sum, \prod\}$ *and* $d \in \{d_H, d_D\}$ *then* WINVER *for* $\nabla_{d,aggr}$ *is in* $\mathbf{P}^{\mathbf{NP}[2]}$*.*

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Conclusions

- We explore complexity bounds for judgment aggregation based on minimization of aggregate distance
- Winner set verification for typical distance-based rules is NP-complete or slightly harder (couldn't be easier!)
- In the general case, the complexity can be as wild as you like (=undecidable)
- Standard structural conditions on distance and aggregation functions are not enough to tame complexity – constraints on computability were needed

Thank you for your attention!