Introduction to Tournaments

Stéphane Airiau

ILLC

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- Voting
  - Input: Preference of agents over a set of candidates or outcomes
  - Output: one candidate or outcome (or a set)
- Tournament
  - Input: Binary relation between outcomes or candidates
  - Output: One candidate or outcome (or a set)

When no ties are allowed between any two alternatives.
Either x beats y or y beats x.

which are the best outcomes?

Notations

- X is a *finite* set of alternatives.
- T is a relation on X, i.e, $T \subset X^2$.
- notation: $(x, y) \in T \Leftrightarrow xTy \Leftrightarrow x \to y \Leftrightarrow x$ "beats" y
- $\mathscr{T}(X)$ is the set of tournaments on X
- $T^+(x) = \{y \in X \mid xTy\}$: successors of x.
- $T^-(x) = \{y \in X \mid yTx\}$: predessors of x.
- $s(x) = \#T^+(x)$ is the Copeland score of x.

Definition (Tournament)

The relation T is a tournament iff

1. $\forall x \in X \ (x, x) \notin T$
2. $\forall (x, y) \in X^2 \ x \neq y \Rightarrow [((x, y) \in T) \vee ((y, x) \in T)]$
3. $\forall (x, y) \in X^2 \ (x, y) \in T \Rightarrow (y, x) \notin T$.

A tournament is a complete and asymmetric binary relation

Majority voting and tournament:

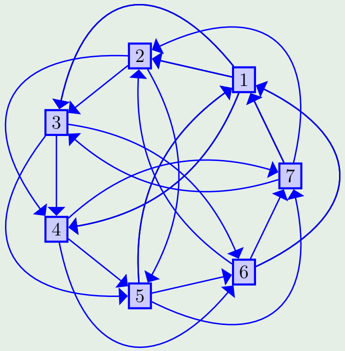
• I finite set of individuals. The preference of an individual i is represented by a complete order P_i defined on X.

• The outcome of majority voting is the binary relation $M(P)$ on X such that $\forall (x, y) \in X$, $xM(P)y \Leftrightarrow \#\{i \in I | xP_iy\} > \#\{i \in I | yP_ix\}$

If initial preferences are strict and number of individual is odd, $M(P)$ is a tournament.

Example (cyclone of order n)

Z_n set of integers modulo n.
$xC_ny \Leftrightarrow y - x \in \{1, \ldots, \frac{n-1}{2}\}$
$T^+(1) = \{2, 3, 4\}$
$T^-(1) = \{5, 6, 7\}$



Definition (isomorphism)

Let X and Y be two sets, $T \in \mathscr{T}(X)$, $U \in \mathscr{T}(Y)$ two tournaments on X and Y.
A mapping $\phi : X \to Y$ is a tournament isomorphism iff

- ϕ is a bijection
- $\forall (x, y) \in X^2$, $xTx' \Leftrightarrow \phi(x)U\phi(x')$

On a set X of cardinal n, there are $2^{\frac{n \cdot (n-1)}{2}}$ tournaments, but many of them are isomorphic.

n	$2^{\frac{n(n-1)}{2}}$	number of non-isomorphic tournaments
8	268,435,456	6,880
10	35,184,372,088,832	9,733,056

Outline

Condorcet principle

Definition (Condorcet winners)

Let $T \in \mathscr{T}(X)$. The set of Condorcet winners of T is

$$\mathscr{C}ondorcet(T) = \{x \in X \mid \forall y \in X, y \neq x \Rightarrow xTy\}$$

Property

Either $\mathscr{C}ondorcet(T) = \emptyset$ or $\mathscr{C}ondorcet(T)$ is a singleton.

Definition (Tournament solution)

A tournament solution $\mathscr{S}$ associates to any tournament $\mathscr{T}(X)$ a subset $\mathscr{S}(T) \subset X$ and satisfies

- $\forall T \in \mathscr{T}(X)$, $\mathscr{S}(T) \neq \emptyset$
- For any tournament isomorphism ϕ, $\phi o \mathscr{S} = \mathscr{S} o \phi$ (anonymity)
- $\forall T \in \mathscr{T}(X)$, $\mathscr{Condorcet}(T) \neq \emptyset \Rightarrow \mathscr{S}(T) = \mathscr{Condorcet}(T)$

For $\mathscr{S}$, $\mathscr{S}_1$, $\mathscr{S}_2$ tournament solutions.

- $\mathscr{S}_1 o \mathscr{S}_2(T) = \mathscr{S}_1(T/\mathscr{S}_2(T)) = \mathscr{S}_1(\mathscr{S}_2(T))$
- $\mathscr{S}^1 = \mathscr{S}$, $\mathscr{S}^{k+1} = \mathscr{S} o \mathscr{S}^k$, $\mathscr{S}^\infty = \lim_{k\to\infty} \mathscr{S}^k$
- solutions may be finer/more selective:
  $\mathscr{S}_1 \subset \mathscr{S}_2 \Leftrightarrow \forall T \in \mathscr{T}(X)\ \mathscr{S}_1(T) \subset \mathscr{S}_2(T)$ than $\mathscr{S}_2$.
- solutions may be different:
  $\mathscr{S}_1 \varnothing \mathscr{S}_2 \Leftrightarrow \exists T \in \mathscr{T} \mid \mathscr{S}_1(T) \cap \mathscr{S}_2(T) = \emptyset$
- solution may have common elements:
  $\mathscr{S}_1 \cap \mathscr{S}_2 \Leftrightarrow \forall T \in \mathscr{T} \mid \mathscr{S}_1(T) \cap \mathscr{S}_2(T) \neq \emptyset$

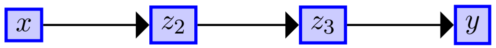
A first solution: the Top Cycle (TC)

Definition (Top Cycle)

The top cycle of $T \in \mathscr{T}(X)$ is the set TC defined as

$$TC(T) = \left\{ x \in X \mid \forall y \in X, \exists k > 0 \;\middle|\; \begin{array}{l} \exists (z_1, \ldots, z_k) \in X^k, \\ z_1 = x,\ z_k = y, \\ \text{and} \\ 1 \leq i < j \leq k \Rightarrow z_i T z_j \end{array} \right\}$$

The top cycle contains outcomes that beat directly or indirectly every other outcomes.



Properties of Solutions

- Regular
- Monotonous
- Independent of the losers
- Strong Superset Property
- Idempotent
- Aïzerman property
- Composition-consistent and weak composition-consistent

Definition (Regular tournament)

A tournament is regular iff all the points have the same Copeland score.

Definition (Monotonous)

A solution $\mathscr{S}$ is monotonous iff $\forall T \in \mathscr{T}(X)$, $\forall x \in \mathscr{S}(T)$, $\forall T' \in \mathscr{T}(X)$ such that $\left\{ \begin{array}{l} T'/X \setminus \{x\} = T/X \setminus \{x\} \\ \forall y \in X,\ xTY \Rightarrow xT'y \end{array} \right.$
one has $x \in \mathscr{S}(T')$

“Whenever a winner is reinforced, it does not become a loser.”

Definition (Independence of the losers)

A solution $\mathscr{S}$ is independent of the losers iff $\forall T \in \mathscr{T}(X)$, $\forall T' \in \mathscr{T}(X)$ such that $\forall x \in \mathscr{S}(T)$, $\forall y \in X$, $xTy \Leftrightarrow xT'y$ one has $\mathscr{S}(T) = \mathscr{S}(T')$.

> *"the only important relations are* $\left\{ \begin{array}{l} \textit{winners to winners} \\ \textit{winners to losers} \end{array} \right.$ *"*
>
> *"What happens between losers do not matter."*

Definition (Strong Superset Property (SSP))

A solution $\mathscr{S}$ satisfies the Strong Superset Property (SSP) iff $\forall T \in \mathscr{T}(X)$, $\forall Y \mid \mathscr{S}(T) \subset Y \subset X$ one has $\mathscr{S}(T) = \mathscr{S}(T/Y)$

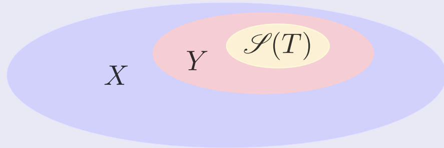
> *"We can delete some or all losers, and the set of winners does not change"*

Definition (Idempotent)

A solution $\mathscr{S}$ is idempotent iff $\mathscr{S} o \mathscr{S} = \mathscr{S}$.

Definition (Aïzerman property)

A solution $\mathscr{S}$ satisfies the Aïzerman property iff $\forall T \in \mathscr{T}(X)$, $\forall Y \subset X$ $\mathscr{S}(T) \subset Y \subset X \Rightarrow \mathscr{S}(T/Y) \subset \mathscr{S}(T)$



Solution Concepts

- Copeland solution (C)
- the Long Path (LP)
- Markov solution (MA)
- Slater solution (SL)

method for ranking

- Uncovered set (UC)
- Iterations of the Uncovered set (UC^∞)
- Dutta's minimal covering set (MC)

based on the notion of covering

- Bipartisan set (BP)

Game theory based

- Bank's solution (B)
- Tournament equilibrium set (TEQ)

Based on Contestation

	TC	UC	UC$^\infty$	MC	BP	B	TEQ	SL	C
Monotonicity	✓	✓	✗	✓	✓	✓	?	✓	✓
Independence of the losers	✓	✗	✗	✓	✓	✗	?	✗	✗
Idempotency	✓	✗	✓	✓	✓	✗	?	✗	✗
Aïzerman property	✓	✓	✗	✓	✓	✓	?	✗	✗
Strong superset property	✓	✗	✗	✓	✓	✗	?	✗	✗
Composition-consistency	✗	✓	✓	✓	✓	✓	✓	✗	✗
Weak Comp.-consist.	✓	✓	✓	✓	✓	✓	✓	✓	✗
Regularity	✓	✓	✓	✓	✓	✗	✗	✓	✗
Copeland value	1	1	1/2	1/2	1/2	$\leq 1/3$	$\leq 1/3$	1/2	1
Complexity	$O(n^2)$	$O(n^{2.38})$	$\mathcal{P}$			$\mathcal{NP}$-hard	$\mathcal{NP}$-hard	$\mathcal{NP}$-hard	$O(n^2)$

	TC	UC	UC$^\infty$	MC	BP	B	TEQ	C
UC	$\subset$							
UC$^\infty$	$\subset$	$\subset$						
MC	$\subset$	$\subset$	$\subset$					
BP	$\subset$	$\subset$	$\subset$	$\subset$				
B	$\subset$	$\subset$	$\cap$	$\cap$	a			
TEQ	$\subset$	$\subset$	$\subset$	b	a	$\subset$		
C	$\subset$	$\subset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	
SL	$\subset$	$\subset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

a $\exists T \in \mathscr{T}_{29} \mid B(T) \subset BP(T)$ and $B(T) \neq BP(T)$
$\exists T' \in \mathscr{T}_6 \mid BP(T') \subset B(T')$ and $B(T') \neq BP(T')$.
It is unknown if $B \cap BP$ can be empty.
Same for TEQ and BP.

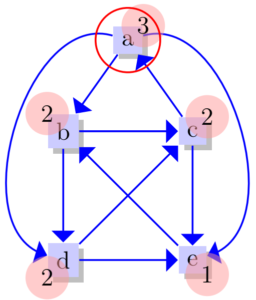
b TEQ $\subset$ MC is a conjecture

Outline

Recall: Copeland score $s(x) = |T^+(x)| = |\{y \in X \mid xTy\}|$
$s(x)$ is the number of alternatives that x beats.

Definition (Copeland solution (C))

Copeland winners of $T \in \mathscr{T}(X)$ is
$C(T) = \{x \in X \mid \forall y \in X,\ s(y) = s(x)\}$



Definition (Slater, Kandall, or Hamming distance)

Let $(T, T') \in \mathscr{T}(X)$
$\Delta(T, T') = \frac{1}{2} \# \left\{(x, y) \in X^2 \mid xTy \wedge yT'x\right\}$

How many arrows are flipped in the tournament graph?

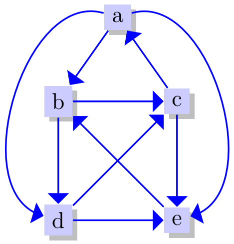
Definition (Slater order)

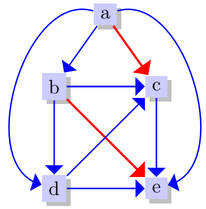
Let $T \in \mathscr{T}(X)$.
A Slater order for T is a **linear order** $U \in \mathscr{L}(X)$ such that

$$\Delta(T, U) = \min_{V \in \mathscr{L}(X)} \{\Delta(T, V)\}$$

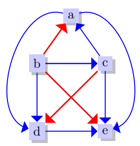
where $\mathscr{L}(X)$ is the set of linear order over X.
The set of Slater winners of T, noted $SL(T)$, is the set of alternatives in X that are Condorcet winner of a Slater order for T.

idea: approximate the tournament by a linear order.

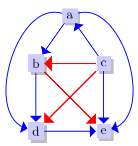




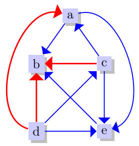
$a \succ b \succ d \succ c \succ e$



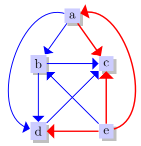
$b \succ c \succ a \succ d \succ e$



$c \succ a \succ b \succ d \succ e$



$d \succ c \succ a \succ e \succ b$



$e \succ a \succ b \succ d \succ c$

to make b, c, d a Condorcet winner, it needs "3 flips"
to make e a Condorcet winner, it needs "4 flips"

Theorem

Computing a Slater ranking is $\mathcal{NP}$-hard.

Noga Alon. Ranking tournaments. *SIAM Journal of Discrete Mathematics*, 20(1):137-142, 2006

Vincent Conitzer, Computing Slater Rankings using similarities among candidates, AAAI, 2006

Outline

Definition (Covering)

Let $T \in \mathscr{T}(X)$ and $(x, y) \in X^2$
x covers y in X iff $[xTy$ and $(\forall z \in X, yTz \Rightarrow xTz)]$
We note $x \rhd y$

Definition (Equivalent definition of covering)

- $x \rhd y$ iff xTy and $\forall z \in X$, $T/_{\{x,y,z\}}$ is transitive.
- $x \rhd y$ iff $x \neq y$ and $T^+(y) \subset T^+(x)$
- $x \rhd y$ iff $x \neq y$ and $T^-(x) \subset T^-(y)$

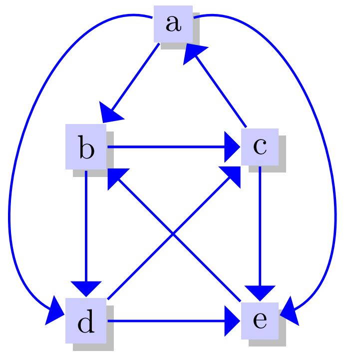
Definition (Uncovered Set (UC))

The uncovered set of T is $UC(T) = \{x \in X \mid \nexists y \in X \mid y \rhd x\}$

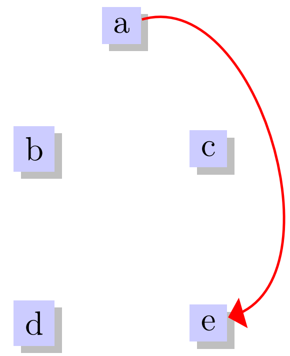
Miller. Graph Theoretical approaches to the Theory of Voting. *American Journal of Political Sciences*, 21:769-803, 1977

Fishburn. Condorcet social choice functions. *SIAM Journal of Applied Mathematics*, 33:469–489, 1977

Any outcome x in the Uncovered Set either beats y, or beats some z that beats y (x beats any other outcome it at most two steps).



tournament



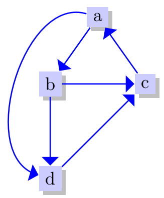
covering relation $\rhd$

$UC(T) = \{a, b, c, d\}$

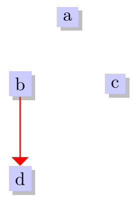
Proposition

$\forall x \in X \setminus UC(X)$, $UC^\infty(X) = UC^\infty(X \setminus \{x\})$

Find a covered alternative, remove it, continue...

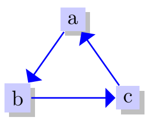


$T/_{\{a,b,c,d\}}$

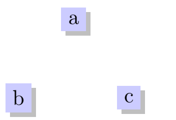


covering relation $\rhd$

$UC(T/_{\{a,b,c,d\}}) = \{a, b, c\}$



$T/_{\{a,b,c\}}$



covering relation $\rhd$

Definition (Covering set)

Let $T \in \mathscr{T}(X)$ and $Y \subset X$.
Y is a Covering set for T iff $\forall x \in X \setminus Y$, $x \notin UC(Y \cup \{x\})$.
(x is covered by some elements in Y)
$C(T)$ is the family of covering sets for T.

Proposition

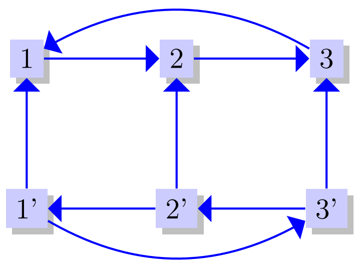
$\forall k \in (\mathbb{N} \cup \infty)$, $UC^k(T)$ is a covering set for T.

proposition

The family C(T) admits a minimal element (by inclusion) called the minimal covering set of T and denoted by $MC(T)$.

Dutta B. Covering sets and a new Condorcet choice correspondence. *Journal of Economic Theory 44(1):63-80, 1988*

$MC \subset UC^\infty$ and $MC \neq UC^\infty$



$$UC(T) = X = UC^\infty(T)$$
$$MC(T) = \{1, 2, 3\}$$

Outline

Definition (tournament game)

A tournament game is a finite symmetric two-player game (X, g) such that, $\forall (x, y) \in X^2$

- $g(x, y) + g(y, x) = 0$ (zero-sum game)
- $x \neq y \Rightarrow g(x, y) \in \{-1, 1\}$

$T \in \mathscr{T}(X) \leftrightarrow$ tournament game (X, g)
with $\forall (x, y) \in X^2$, xTy iff $g(x, y) = +1$

Propositions

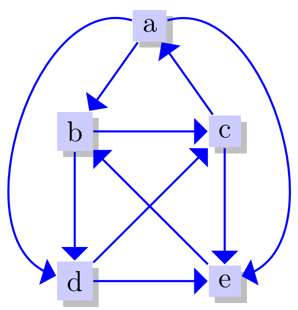
- y is a Condorcet winner $\Rightarrow \forall x \in X$, y is a best response to x.
- y is not a Condorcet winner $\Rightarrow \forall x \mid xTy$, x is a best response to y.
- (x, y) is a pure Nash equilibrium iff $\begin{cases} x = y \\ x \text{ is a Condorcet winner} \end{cases}$
- x dominates y in $(X, g) \Leftrightarrow x$ covers y
  - $UC(T)$ is the set of undominated strategies
  - $UC^\infty(T)$ is the set of strategies not sequentially dominated.

Theorem

A tournament game has a unique Nash equilibrium in mixed strategy, and this equilibrium is symmetric.

Definition (Bipartisan Set)

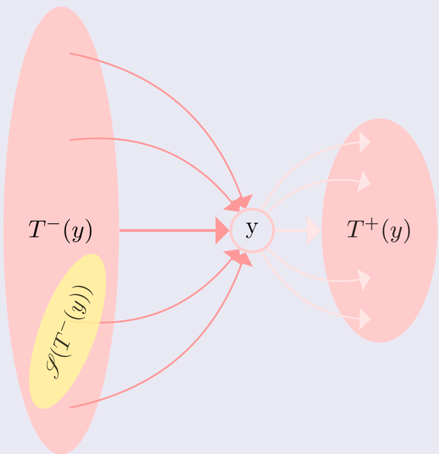
Let $T \in \mathscr{T}(X)$.
The Bipartisan set $BP(X)$ is the support of the unique mixed equilibrium of the tournament game associated with T.



↱	a	b	c	d	e
a	0	1	-1	1	1
b	-1	0	1	1	-1
c	1	-1	0	-1	1
d	-1	-1	1	0	1
e	-1	1	-1	-1	0

Outline

Is y a good outcome?



For a solution tournament $\mathscr{S}$ and $T \in \mathscr{T}(X)$,
$\forall (x,y) \in X^2 \; xD(\mathscr{S},T)y \Leftrightarrow x \in S(T \,|\, T^-(y))$
x is a contestation of y for T according to $\mathscr{S}$.

Bank's set

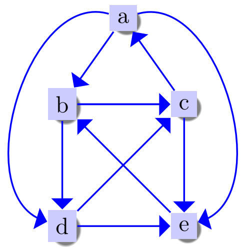
There exists a unique tournament solution B such that

$$\forall T \in \mathscr{T}(X),\, o(T) \geq 2 \Rightarrow B(T) = D(B,T)^-(X)$$

$D(B,T)^-(X)$ is the set of points in X which are contestation of some point of X according to $\mathscr{S}$.

Proposition

$x \in B(T)$ iff $\exists Y \subset X$ such that $x \in Y$ and $T|Y$ i an ordering for which x is the winner and no point of X beats all the points of Y.



a $Y = \{d\}$, $a \succ d$ and aTb, dTc, aTe. ✓

b $Y = \{d, c\}$, $b \succ d \succ c$ and cTa, cTe. ✓

c $Y = \{a\}$, $c \succ a$ and aTb, aTd, aTe. ✓

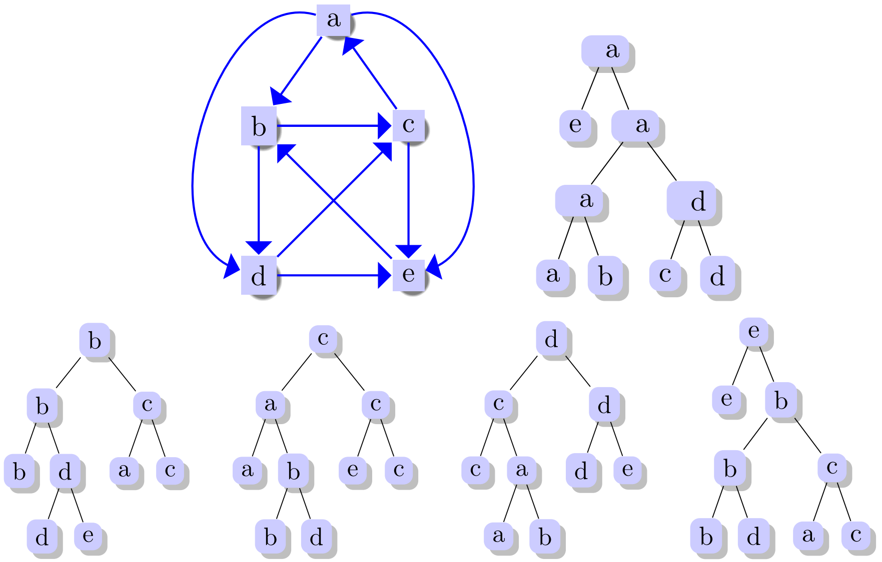
d $Y = \{c, e\}$, $d \succ c \succ e$ and cTa, eTb. ✓

e $Y = \{b\}$ no because of aTb and aTe.

$Y = \{b, c\}$ not an ordering. ✗

$B(T) = \{a, b, c, d\}$

Outline



Definition (Algebraic solution)

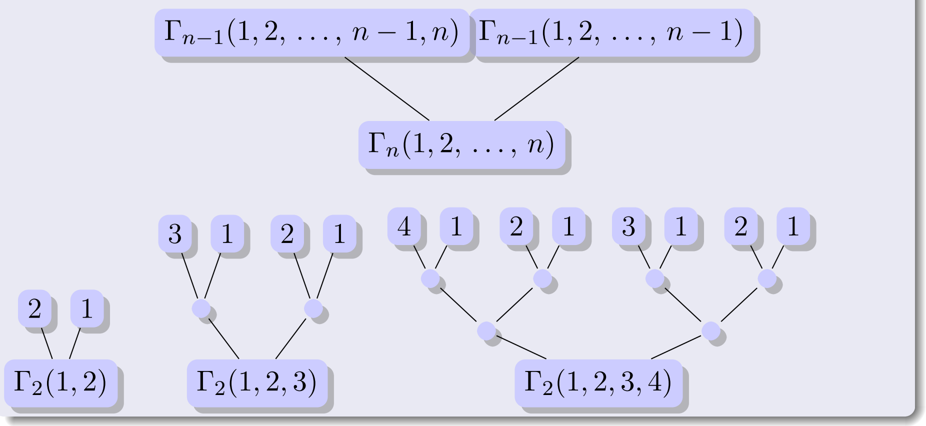
A tournament solution $\mathscr{S}$ is computable by a binary tree if, for any order n, there exists a labelled binary tree (N, A, i) of order n such that, for any tournament $T \in \mathscr{T}(X)$ of order n, $\mathscr{S}(T)$ is the set of winners of T along (N, T, i) for all drawing of X.
$\mathscr{S}$ is computable by a binary tree iff $\mathscr{S}$ is algebraic.

- Any algebraic tournament solution selects a winner in the top cycle.
- The Copeland and Markov solutions are not algebraic.
- Strengthening a winner can make her lose.
- There exists a non monotonous algebraic tournament solution.

Miller. Graph Theoretical approaches to the Theory of Voting. *American Journal of Political Sciences*, 21:769-803,1977

McKelvey, Niemi. A multistage game representation of sophisticated voting for binary procedures. *Journal of Economic Theory* 18:1-22,1978

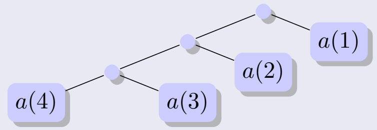
Multistage elimination tree or sophisticated agenda



Miller. Graph Theoretical approaches to the Theory of Voting. *American Journal of Political Sciences*, 21:769-803,1977

Hervé Moulin. Dominance Solvable Voting Schemes, *Econometrica*, 47(6):1337-1352,1979

Sophisticated voting on simple agendas



- $\Gamma_k(a)$: outcome of *strategic* voting on the simple agenda of order k with agenda a
- $a_{-n} = a(1) \cdot a(2) \ldots a(n-2) \cdot a(n-1)$
- $a_{-(n-1)} = a(1) \cdot a(2) \ldots a(n-2) \cdot a(n) \ldots a(n)$

Voting for $a(n)$ or $a(n-1) \Rightarrow$ Comparing $\Gamma_{n-1}(a_{-n})$ and $\Gamma_{n-1}(a_{-(n-1)})$, i.e.,

$$\Gamma_n(a) = \Gamma_{n-1}(a_{-n}) \cdot \Gamma_{n-1}(a_{-(n-1)})$$

Sophisticated agenda and sophisticated voting

Strategic voting one a simple agenda results in choosing the winner of the associated sophisticated agenda.

Property

Let $\mathcal{B}$ the set of all permutations of $X = \{1,\ldots,n\}$
Let $a \in \mathcal{B}$, $w(\Gamma_n, T, a)$ is the winner of the tournament $T \in \mathscr{T}(X)$ along the sophisticated agenda Γ_n for the drawing a.

$$\{w(\Gamma_n, T, a), a \in \mathcal{B}\} = Bank(T)$$

Knockout tournaments

Definition (General Knockout Tournament)

Given a set N of players and a matrix P such that P_{ij} denotes the probability that player i wins against player j in a pairwise elimination match and $\forall (i,j) \in N^2$ $0 \le P_{ij} = 1 - P_{ji} \le 1$,
a knockout tournament $KTN = (T, S)$ is defined by:

- A tournament structure T: a binary tree with |N| leaf nodes
- A seeding S: a bijection between the players in N and the leaf nodes of T

Theorem

It is $\mathcal{NP}$-complete to decide whether there exists a tournament structure KT with round placement R such that a target player $k \in N$ will win the tournament.

Thuc Vu, Alon Altman, Yoav Shoham, *"On the Complexity of Schedule Control Problems for Knockout Tournaments"*, AAMAS 2009

Outline

Properties

For Bipartisan set, minimal covering set, iterated uncovered set and the top cycle

- if $\exists$ a Condorcet winner, the winner is unique (definition)
- if $\nexists$ a Condorcet winner, the set of winners contains at least 3 alternatives.

Properties

If all tournaments are equiprobable, the top cycle is almost surely the whole set of alternatives.
Probability that every alternative is in the Banks set in a random tournament goes to one as the number of alternatives goes to infinity. (*every* alternative is in the Banks set in *almost all* tournaments).

Mark Fey. Choosing from a large tournament, *Social Choice and Welfare, 31(2):301–309*

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- F. Brandt, F. Fischer, P. Harrenstein, and M. Mair. *“A computational analysis of the tournament equilibrium set”*. AAAI-2008, COMSOC-2008.