Computational Social Choice: Spring 2009

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Plan for Today

This lecture will be an introduction to voting theory. Voting is the most obvious mechanism by which to come to a collective decision, so it is a central topic in social choice theory. Topics today:

- many *voting procedures:* e.g. plurality rule, Borda count, approval voting, single transferable vote, ...
- several (desirable) *properties* of voting procedures: e.g. anonymity, neutrality, monotonicity, strategy-proofness, ...
- some voting *paradoxes*, highlighting that there seems to be no perfect voting procedure

Most of the material on these slides is taken from a review article by Brams and Fishburn (2002).

S.J. Brams and P.C. Fishburn. *Voting Procedures.* In K.J. Arrow *et al.* (eds.), *Handbook of Social Choice and Welfare*, Elsevier, 2002.

Voting Procedures

We'll discuss *voting procedures* for selecting a *single winner* from a finite set of *candidates* (the number of candidates is m).

- A *voter* votes by submitting a *ballot*, e.g., the name of a single candidate, a ranking of all the candidates, or something else.
- The procedure defines what are *valid ballots*, and how to *aggregate* the ballot information to obtain a winner.

Remark I: For all of the procedures to be discussed two or more candidates can come out on top (even if this is unlikely for large numbers of voters). A complete system also has to specify how to deal with such ties, but here we ignore the issue of *tie-breaking*.

Remark II: Formally, *voting rules* map ballots to single winners; *voting correspondences* map ballots to sets of winners.

Plurality Rule

Under the *plurality rule* (a.k.a. *simple majority*), each voter submits a ballot showing the name of one of the candidates standing. The candidate receiving the most votes wins.

This is the most widely used voting procedure in practice.

If there are only two candidates, then it is a very good procedure.

Criticism of the Plurality Rule

Problems with the plurality rule (for more than two candidates):

- The information on voter preferences other than who their favourite candidate is gets ignored.
- Dispersion of votes across ideologically similar candidates.
- Encourages voters not to vote for their true favourite, if that candidate is perceived to have little chance of winning.

Plurality with Run-Off

In the *plurality rule with run-off*, first each voter votes for one candidate. The winner is elected in a second round by using the plurality rule with the two top candidates from the first round.

Used to elect the president in France (and heavily criticised after Le Pen came in second in the first round in 2002).

The No-Show Paradox

Under plurality with run-off, it may be better to abstain than to vote for your favourite candidate! Example:

$$\begin{array}{ll} \text{25 voters:} & A \succ B \succ C \\ \text{46 voters:} & C \succ A \succ B \\ \text{24 voters:} & B \succ C \succ A \end{array}$$

Given these voter preferences, B gets eliminated in the first round, and C beats A 70:25 in the run-off.

Now suppose two voters from the first group abstain:

$$\begin{array}{ll} \text{23 voters:} & A \succ B \succ C \\ \text{46 voters:} & C \succ A \succ B \\ \text{24 voters:} & B \succ C \succ A \end{array}$$

A gets eliminated, and B beats C 47:46 in the run-off.

Monotonicity

We would like a voting procedure to satisfy *monotonicity:* if a particular candidate wins and a voter raises that candidate in their ballot (whatever that means exactly for different sorts of ballots), then that candidate should still win.

The *winner-turns-loser paradox* shows that plurality with run-off does *not* satisfy monotonicity:

$$\begin{array}{ll} \text{27 voters:} & A \succ B \succ C \\ \text{42 voters:} & C \succ A \succ B \\ \text{24 voters:} & B \succ C \succ A \end{array}$$

B is eliminated in the first round and C beats A 66:27 in the run-off. But if 4 of the voters in the first group *raise C to the top* (i.e., join the second group), then B will win (it's the same example as on the previous slide).

Anonymity and Neutrality

On the positive side, both variants of the plurality rule satisfy two important properties:

- *Anonymity:* A voting procedure is anonymous if all *voters* are treated the same: if two voters switch ballots, then the election outcome does not change.
- *Neutrality:* A voting procedure is neutral if all *candidates* are treated the same: if the election winner switches names with some other candidate, then that other candidate will win.

Indeed, (almost) all of the procedures we'll discuss satisfy these properties (we'll see one exception where neutrality is violated).

Often the *tie-breaking* rule can be a source of violating either anonymity (e.g., if one voter has the power to break ties) or neutrality (e.g., if the incumbent wins in case of a tie).

May's Theorem

As mentioned before, if there are only two candidates, then the plurality rule is a pretty good rule to use. Specifically:

Theorem 1 (May, 1952) *For two candidates, a voting rule is anonymous, neutral, and monotonic iff it is the plurality rule.*

Remark: In these slides we assume that there are no ties, but May's Theorem also works for an appropriate definition of monotonicity when ties are possible.

K.O. May. A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decisions. *Econometrica*, 20(4):680–684, 1952.

Proof Sketch

Clearly, plurality does satisfy all three properties. ✓

Now for the other direction:

For simplicity, assume the number of voters is odd (no ties).
Plurality-style ballots are fully expressive for two candidates.
Anonymity and neutrality $\rightsquigarrow$ only number of votes matters.

Denote as A the set of voters voting for candidate a and as B those voting for b. Distinguish two cases:

- Whenever $|A| = |B| + 1$ then a wins. Then, by monotonicity, a wins whenever $|A| > |B|$ (that is, we have plurality). ✓
- There exist A, B with $|A| = |B| + 1$ but b wins. Now suppose one a-voter switches to b. By monotonicity, b still wins. But now $|B'| = |A'| + 1$, which is symmetric to the earlier situation, so by neutrality a should win $\rightsquigarrow$ contradiction. ✓

Borda Rule

Under the voting procedure proposed by Jean-Charles de Borda, each voter submits a complete ranking of all m candidates.

For each voter that places a candidate first, that candidate receives $m-1$ points, for each voter that places her 2nd she receives $m-2$ points, and so forth. The *Borda count* is the sum of all the points.

The candidate with the highest Borda count wins.

This takes care of some of the problems identified for plurality voting. For instance, this form of balloting is more informative.

A disadvantage (of any system requiring voters to submit full rankings) are the high *elicitation* and *communication* costs.

J.-C. de Borda. *Mémoire sur les élections au scrutin*. Histoire de l'Académie Royale des Sciences, Paris, 1781.

Pareto Principle

A voting procedure satisfies the (weak) *Pareto principle* if, whenever candidate A is (strictly) preferred over candidate B by all voters, then B cannot win the election.

Clearly, both the plurality rule and the Borda rule satisfy the Pareto principle.

Positional Scoring Rules

We can generalise the idea underlying the Borda count as follows:

Let m be the number of candidates. A *positional scoring rule* is given by a *scoring vector* $s = \langle s_1, \ldots, s_m \rangle$ with $s_1 \geq s_2 \geq \cdots \geq s_m$.

Each voter submits a ranking of all candidates. Each candidate receives s_i points for every voter putting her at the ith position. The candidate with the highest score (sum of points) wins.

- The *Borda rule* is is the positional scoring rule with the scoring vector $\langle m{-}1, m{-}2, \ldots, 0 \rangle$.
- The *plurality rule* is the positional scoring rule with the scoring vector $\langle 1, 0, \ldots, 0 \rangle$.

Condorcet Principle

Recall the Condorcet Paradox (first lecture):

$$\begin{array}{ll} \text{Voter 1:} & A \succ B \succ C \\ \text{Voter 2:} & B \succ C \succ A \\ \text{Voter 3:} & C \succ A \succ B \end{array}$$

A majority prefers A over B and a majority also prefers B over C, but then again a majority prefers C over A. Hence, no single candidate would beat any other candidate in pairwise comparisons.

In cases where the *is* such a candidate beating everyone else in a pairwise majority contest, we call her the *Condorcet winner*.

Observe that if there is a Condorcet winner, then it must be unique.

A voting procedure is said to satisfy the *Condorcet principle* if it elects the Condorcet winner whenever there is one.

Positional Soring violates Condorcet

Consider the following example:

$$\begin{array}{ll} \text{3 voters:} & A \succ B \succ C \\ \text{2 voters:} & B \succ C \succ A \\ \text{1 voter:} & B \succ A \succ C \\ \text{1 voter:} & C \succ A \succ B \end{array}$$

A is the *Condorcet winner*; she beats both B and C 4:3. But any *positional scoring rule* assigning strictly more points to a candidate placed 2nd than to a candidate placed 3rd ($s_2 > s_3$) makes B win:

$$\begin{array}{ll} A: & 3 \cdot s_1 + 2 \cdot s_2 + 2 \cdot s_3 \\ B: & 3 \cdot s_1 + 3 \cdot s_2 + 1 \cdot s_3 \\ C: & 1 \cdot s_1 + 2 \cdot s_2 + 4 \cdot s_3 \end{array}$$

This shows that *no positional scoring rule* (with a strictly descending scoring vector) will satisfy the *Condorcet principle*.

Copeland Rule

Some voting procedures have been designed specifically to meet the Condorcet principle.

The *Copeland rule* elects a candidate that maximises the difference between won and lost pairwise majority contests.

The Copeland rule satisfies the Condorcet principle.

Dodgson Rule

Charles L. Dodgson (a.k.a. Lewis Carroll of "Alice in Wonderland" fame) proposed a voting method that selects the candidate minimising the number of "switches" in the voters' linear preference orderings required to make that candidate a Condorcet winner.

Clearly, this metric is 0 if the candidate in question already is a Condorcet winner, so the *Dodgson rule* certainly satisfies the Condorcet principle.

C.L. Dodgson. *A Method of Taking Votes on more than two Issues.* Clarendon Press, Oxford, 1876.

Approval Voting

In *approval voting*, a ballot may consist of any subset of the set of candidates. These are the candidates the voter approves of. The candidate receiving the most approvals wins.

Approval voting has been used by several professional societies, such as the American Mathematical Society (AMS).

Intuitive advantages of approval voting include:

- No need *not* to vote for the most preferred candidate for strategic reasons when she has a slim chance of winning.
- Good compromise between plurality (too simple) and Borda (too complex) in terms of communication requirements.

Single Transferable Vote (STV)

Also known as the *Hare system*. To select a single winner, it works as follows (voters submit ranked preferences for all candidates):

- If one of the candidates is the 1st choice for over 50% of the voters (*quota*), she wins.
- Otherwise, the candidate who is ranked 1st by the fewest voters gets *eliminated* from the race.
- Votes for eliminated candidates get *transferred:* delete removed candidates from ballots and "shift" rankings (i.e., if your 1st choice got eliminated, then your 2nd choice becomes 1st).

In practice, voters need not be required to rank all candidates (non-ranked candidates are assumed to be ranked lowest).

STV (suitably generalised) is often used to elect committees.

STV is used in several countries (e.g., Australia, New Zealand, ...).

Example

Elect one winner amongst four candidates, using STV (100 voters):

$$\begin{array}{ll} \text{39 voters:} & A \succ B \succ C \succ D \\ \text{20 voters:} & B \succ A \succ C \succ D \\ \text{20 voters:} & B \succ C \succ A \succ D \\ \text{11 voters:} & C \succ B \succ A \succ D \\ \text{10 voters:} & D \succ A \succ B \succ C \end{array}$$

(Answer: B wins)

Note that for 3 candidates, STV reduces to plurality voting with run-off, so it suffers from the same paradoxes.

Manipulation: Plurality Rule

Suppose the *plurality rule* (as in most real-world situations) is used to decide the outcome of an election.

Assume the preferences of the people in, say, Florida are as follows:

$$\begin{array}{ll} 49\%: & \text{Bush} \succ \text{Gore} \succ \text{Nader} \\ 20\%: & \text{Gore} \succ \text{Nader} \succ \text{Bush} \\ 20\%: & \text{Gore} \succ \text{Bush} \succ \text{Nader} \\ 11\%: & \text{Nader} \succ \text{Gore} \succ \text{Bush} \end{array}$$

So even if nobody is cheating, Bush will win in a plurality contest.

It would have been in the interest of the Nader supporters to *manipulate*, i.e., to misrepresent their preferences.

The Gibbard-Satterthwaite Theorem

The Gibbard-Satterthwaite Theorem is widely regarded as *the* central result in voting theory. Broadly, it states that there can be no "reasonable" voting rule that would not be manipulable.

Our formal statement of the theorem follows Barberà (1983). We won't prove it here. A proof that is similar to the one we have discussed for Arrow's Theorem is given by Benoît (2000).

A. Gibbard. Manipulation of Voting Schemes: A General Result. *Econometrica*, 41(4):587–601, 1973.

M.A. Satterthwaite. Strategy-proofness and Arrow's Conditions. *Journal of Economic Theory*, 10:187–217, 1975.

S. Barberà. Strategy-proofness and Pivotal Voters: A Direct Proof of the Gibbard-Satterthwaite Theorem. *Intl. Economic Review*, 24(2):413–417, 1983.

J.-P. Benoît. The Gibbard-Satterthwaite Theorem: A Simple Proof. *Economic Letters*, 69:319–322, 2000.

Setting and Notation

- Finite set A of *candidates* (alternatives); finite set $I = \{1, \ldots, n\}$ of *voters* (individuals).
- A *preference ordering* is a strict linear order on A. The set of all such orderings is denoted $\mathcal{P}$. Each voter i has an *individual* preference ordering P_i. A *preference profile* $\langle P_1, \ldots, P_n \rangle \in \mathcal{P}^n$ consists of a preference ordering for each voter.
- The *top candidate* $top(P)$ of a preference ordering P is defined as the unique $x \in A$ such that xPy for all $y \in A \setminus \{x\}$.
- We write $(\boldsymbol{P}_{-i}, P')$ for the preference profile we obtain when we *replace* P_i by P' in the preference profile $\boldsymbol{P}$.
- A *voting rule* is a function $f : \mathcal{P}^n \to A$ mapping preference profiles to winning candidates (so the P_i are used as ballots).

Statement of the Theorem

A voting rule f is *dictatorial* if the winner is always the top candidate of a particular voter (the dictator):

$$(\exists i \in I)(\forall \boldsymbol{P} \in \mathcal{P}^n)[f(\boldsymbol{P}) = top(P_i)]$$

A voting rule f is *manipulable* if it may give a voter an incentive to misrepresent their preferences:

$$(\exists \boldsymbol{P} \in \mathcal{P}^n)(\exists P' \in \mathcal{P})(\exists i \in I)[f(\boldsymbol{P}_{-i}, P')\, P_i\, f(\boldsymbol{P})]$$

A voting rule that is not manipulable is called *strategy-proof*.

Theorem 2 (Gibbard-Satterthwaite) *Every strategy-proof voting rule for three or more candidates must be dictatorial.*

Remarks: (1) Can be extended to *voting correspondences*, allowing for *sets* of winners (Duggan-Schwartz Theorem). (2) Does *not* apply to *approval voting* (input to f are not linear orders).

Control: Borda Rule

The technical term "manipulation" refers to voters misrepresenting their preferences, but there are also other forms of manipulation ...

Suppose we are using the *Borda rule* to elect one winner from amongst 4 candidates, and there are 13 voters:

$$\begin{array}{ll} \text{4 voters:} & A \succ X \succ B \succ C \\ \text{3 voters:} & C \succ A \succ X \succ B \\ \text{6 voters:} & B \succ C \succ A \succ X \end{array}$$

We get the following Borda scores: A (24), B (22), C (21), X (11).

We may suspect the A-supporters of having nominated X in order to *control* the election. For, without X, we would get the following Borda scores: A (11), B (16), C (12).

Agenda Manipulation: Voting Trees

The term *control* is used for any kind of "manipulation" that involves changing the structure of an election (voting rule, set of candidates, ...). This is typically something that the *election chair* may do (but not only; see nomination example on previous slide).

Consider the following example (Condorcet triple):

$$\begin{array}{ll} \text{Voter 1:} & A \succ B \succ C \\ \text{Voter 2:} & B \succ C \succ A \\ \text{Voter 3:} & C \succ A \succ B \end{array}$$

Suppose the voting rule is given by a *binary tree*, with the candidates labelling the leaves, and a candidate progressing to a parent node if beats its sibling in a *majority contest*.

Then the election chair can influence the election outcome by changing the *agenda* (here, the exact binary tree to be used) ...

Agenda Manipulation: Voting Trees (cont.)

Here are again the voter preferences from the previous slide:

$$
\begin{array}{ll}
\text{Voter 1:} & A \succ B \succ C \\
\text{Voter 2:} & B \succ C \succ A \\
\text{Voter 3:} & C \succ A \succ B
\end{array}
$$

So in a pairwise majority contest, A will beat B; B will beat C; and C will beat A. Here are two possible voting trees:

```
(1)                   (2)       o
                               / \
           o                  /   \
          / \
         o   C               o     o
        / \                 / \   / \
       A   B               A   B B   C
```

If `(1)` is used then C will win; if `(2)` is used then A will win. That is, these voting rules violate *neutrality*.

Classification of Voting Procedures

Brams and Fishburn (2002) list many more voting procedures.

The structure of their paper implicitly suggests a (rough) classification of voting rules:

- Nonranked input: plurality rule, approval voting
- Successive elimination: plurality with run-off, STV, voting trees
- Condorcet procedures: Copeland, Dodgson, (many more)
- Positional scoring rules: Borda count

S.J. Brams and P.C. Fishburn. *Voting Procedures.* In K.J. Arrow *et al.* (eds.), *Handbook of Social Choice and Welfare*, Elsevier, 2002.

Summary

This has been an introduction to voting theory. The main aim has been to show that there are many alternative systems, all with their own flaws and advantages.

- Voting procedures: plurality (with run-off), positional scoring rules, Condorcet procedures, approval, STV, voting trees, ...
- Properties discussed: anonymity, neutrality, monotonicity, Condorcet principle, strategy-proofness, ...
- Cheating can take many forms: manipulation, bribery, control
- May's Theorem and Gibbard-Satterthwaite Theorem

Most of the material on these slides comes from (and much more can be found in) the review article by Brams and Fishburn (2002).

S.J. Brams and P.C. Fishburn. *Voting Procedures.* In K.J. Arrow *et al.* (eds.), *Handbook of Social Choice and Welfare*, Elsevier, 2002.

What next?

This lecture has concentrated on classical topics in voting theory. Next week we are going to discuss *complexity issues* in voting.

Two questions that suggest why this is of interest:

- What if we have found a voting procedure with many wonderful theoretical properties, but actually *computing the winner* using that rule is a *computationally intractable* problem?
- What if *manipulation* is possible (by the Gibbard-Satterthwaite Theorem), but turns out to be *computationally intractable*, so no voter would ever be able to exploit this weakness?