Computational Social Choice 2023

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Plan for Today

We first review a large number of concrete examples for *voting rules* proposed in the literature and used in practice.

To put some order into this large space of rules, we then shall:

- look into approaches to *classifying* voting rules
- review some *axioms* to differentiate between voting rules

For full details see Zwicker (2016).

W.S. Zwicker. Introduction to the Theory of Voting. In F. Brandt et al. (eds.), *Handbook of Computational Social Choice*. CUP, 2016.

The Model

Fix a finite set $A = \{a, b, c, \ldots\}$ of *alternatives*, with $|A| = m \geqslant 2$.

Let $\mathcal{L}(A)$ denote the set of all strict linear orders R on A. We use elements of $\mathcal{L}(A)$ to model (true) *preferences* and (declared) *ballots*.

Each member i of a finite set $N = \{1, \ldots, n\}$ of *voters* supplies us with a ballot R_i, giving rise to a *profile* $\boldsymbol{R} = (R_1, \ldots, R_n) \in \mathcal{L}(A)^n$.

A *voting rule* (or *social choice function*) for N and A selects (ideally) one or (in case of a tie) more winners for every such profile:

$$F : \mathcal{L}(A)^n \to 2^A \setminus \{\emptyset\}$$

If $|F(\boldsymbol{R})| = 1$ for all profiles $\boldsymbol{R}$, then F is called *resolute*.

Most natural voting rules are *irresolute* and have to be paired with a *tie-breaking rule* to always select a unique election winner.

<u>Examples:</u> random tie-breaking, lexicographic tie-breaking

Examples for Voting Rules

Borda | Plurality | Veto | k-Approval | (Approval Voting)
STV | Plurality with Runoff | Coombs | Nanson | Baldwin
Cup Rule | Condorcet | Copeland | Slater | Kemeny | Banks | Schwartz
Dodgson | Young | Ranked Pairs | Schulze | Simpson | Bucklin | Black
(Range Voting) | (Cumulative Voting) | (Majority Judgment)

Exercise: *What is it that the rules in brackets have in common?*

Classifying Voting Rules

How can we put some order into this zoo of voting rules? Attempts:

- important family of *positional scoring rules* (operational definition)
- important family of *Condorcet extensions* (axiomatic perspective)
- classifying rules in terms of the *information* they require

Remark: Some of the rules we saw do not fit into our formal model (different ballot format), so they also do not fit these classifications.

Positional Scoring Rules

We can generalise the idea underlying the Borda rule as follows:

A *positional scoring rule* (PSR) is defined by a so-called *scoring vector* $\boldsymbol{s} = (s_1, \ldots, s_m) \in \mathbb{R}^m$ with $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_m$ and $s_1 > s_m$.

Each voter submits a ranking of the m alternatives. Each alternative receives s_i points for every voter putting it at the ith position.

The alternative(s) with the highest score (sum of points) win(s).

Examples:

- *Borda rule* = PSR with scoring vector $(m{-}1, m{-}2, \ldots, 0)$
- *Plurality rule* = PSR with scoring vector $(1, 0, \ldots, 0)$
- *Veto rule* = PSR with scoring vector $(0, \ldots, 0, -1)$
- For any $k < m$, *k-approval* = PSR with $(\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$

Exercise: *Name the rule induced by $\boldsymbol{s} = (9, 7, 5)$! General idea?*

Condorcet Extensions

An alternative that beats every other alternative in pairwise majority contests is called a *Condorcet winner*. Sometimes there is no CW:

$$a \succ b \succ c$$
$$b \succ c \succ a$$
$$c \succ a \succ b$$

This is the famous *Condorcet Paradox*.

The *Condorcet Principle* says that, if it exists, only the CW should win. Voting rules that satisfy this principle are called *Condorcet extensions*.

Exercise: *Show that Copeland, Kemeny, and the cup rules are CEs.*

Positional Scoring Rules and the Condorcet Principle

Consider this example with three alternatives and seven voters:

3 voters:	$a \succ b \succ c$
2 voters:	$b \succ c \succ a$
1 voter:	$b \succ a \succ c$
1 voter:	$c \succ a \succ b$

So a is the *Condorcet winner*: a beats both b and c (with 4 out of 7). But any *positional scoring rule* makes b win (because $s_1 \geqslant s_2 \geqslant s_3$):

$$a: \quad 3 \cdot s_1 + 2 \cdot s_2 + 2 \cdot s_3$$
$$b: \quad 3 \cdot s_1 + 3 \cdot s_2 + 1 \cdot s_3$$
$$c: \quad 1 \cdot s_1 + 2 \cdot s_2 + 4 \cdot s_3$$

Thus, *no positional scoring rule* for three (or more) alternatives can possibly satisfy the *Condorcet Principle*.

Fishburn's Classification

Can classify voting rules on the basis of the *information* they require. The best known such classification is due to Fishburn (1977):

- *C1:* Winners can be computed from the *majority graph* alone.

  Examples: Copeland, Slater

- *C2:* Winners can be computed from the *weighted majority graph* (but not from the majority graph alone).

  Examples: Kemeny, Ranked Pairs, Borda

- *C3:* All other voting rules.

  Examples: Young, Dodgson, STV

Remark: Fishburn originally intended this for Condorcet extensions only, but the concept also applies to all other voting rules.

P.C. Fishburn. Condorcet Social Choice Functions. *SIAM Journal on Applied Mathematics*, 1977.

The Axiomatic Method

So many voting rules! How do you choose?

Might employ the *axiomatic method* to formulate *normative principles* (a.k.a. *axioms*) and then choose on that basis. Examples:

- *Participation Principle:* It should be in the best interest of voters to participate; voting truthfully should be no worse than abstaining.
- *Pareto Principle:* There should be no alternative that every voter strictly prefers to the alternative selected by the voting rule.
- *Condorcet Principle:* If there is an alternative that is preferred to every other alternative by a majority of voters, then it should win.

Sometimes, we can even fully *characterise* the unique rule that meets our requirements. Next: example for a seminal result of this kind . . .

Axioms: Anonymity and Neutrality

Two basic fairness requirements for a voting rule F:

- F is *anonymous* if $F(R_1, \ldots, R_n) = F(R_{\pi(1)}, \ldots, R_{\pi(n)})$ for any profile $(R_1, \ldots, R_n)$ and any permutation $\pi : N \to N$.
- F is *neutral* if $F(\pi(\boldsymbol{R})) = \pi(F(\boldsymbol{R}))$ for any profile $\boldsymbol{R}$ and any permutation $\pi : A \to A$ (with π extended to profiles and sets of alternatives in the natural manner).

In other words:

- Anonymity is symmetry w.r.t. voters.
- Neutrality is symmetry w.r.t. alternatives.

Consequences of Axioms

For this slide only, let us restrict attention to voting rules for scenarios with just *two voters* $(n = 2)$ and *two alternatives* $(m = 2)$.

Exercise: *Show that there exists no resolute voting rule that is 'fair' in the sense of being both anonymous and neutral.*

Exercise: *But there still are a couple of irresolute voting rules that are both anonymous and neutral. Give some examples!*

Axiom: Positive Responsiveness

Notation: Write $N^{\boldsymbol{R}}_{x \succ y} = \{i \in N \mid (x,y) \in R_i\}$ for the set of voters who rank alternative x above alternative y in profile $\boldsymbol{R}$.

A (not necessarily resolute) voting rule satisfies *positive responsiveness* if, whenever some voter raises a (possibly tied) winner $x^\star$ in her ballot, then $x^\star$ will become the *unique* winner. Formally:

> F is *positively responsive* if $x^\star \in F(\boldsymbol{R})$ implies $\{x^\star\} = F(\boldsymbol{R'})$ for any alternative $x^\star$ and any two *distinct* profiles $\boldsymbol{R}$ and $\boldsymbol{R'}$ s.t. $N^{\boldsymbol{R}}_{x^\star \succ y} \subseteq N^{\boldsymbol{R}'}_{x^\star \succ y}$ and $N^{\boldsymbol{R}}_{y \succ z} = N^{\boldsymbol{R'}}_{y \succ z}$ for all $y, z \in A \setminus \{x^\star\}$.

Thus, this is a monotonicity requirement (we'll see others later on).

May's Theorem

When there are only *two alternatives*, then all the voting rules we have seen coincide with the *simple majority rule*. Good news:

May's Theorem: *A voting rule for two alternatives satisfies the axioms of anonymity, neutrality, and positive responsiveness if and only if that rule is the simple majority rule.*

This provides a good justification for using this rule (arguing in favour of 'majority' directly is harder than arguing for anonymity etc.).

K.O. May. A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decisions. *Econometrica*, 1952.

Proof Sketch

Clearly, the simple majority rule satisfies all three properties. ✓

Now for the other direction:

Assume the number of voters is *odd* $\rightsquigarrow$ no ties. (other case: similar)

There are two possible ballots: $a \succ b$ and $b \succ a$.
Anonymity $\rightsquigarrow$ only *number of ballots* of each type matters.

Consider all possible profiles $\boldsymbol{R}$. Distinguish two cases:

- Whenever $|N^{\boldsymbol{R}}_{a \succ b}| = |N^{\boldsymbol{R}}_{b \succ a}| + 1$, then only a wins.
  By *PR*, a wins whenever $|N^{\boldsymbol{R}}_{a \succ b}| > |N^{\boldsymbol{R}}_{b \succ a}|$. By *neutrality*, b wins otherwise. But this is just what the simple majority rule does. ✓
- There exist a profile $\boldsymbol{R}$ with $|N^{\boldsymbol{R}}_{a \succ b}| = |N^{\boldsymbol{R}}_{b \succ a}| + 1$, yet b wins.
  Suppose one a-voter switches to b, yielding $\boldsymbol{R'}$. By *PR*, now only b wins. But now $|N^{\boldsymbol{R'}}_{b \succ a}| = |N^{\boldsymbol{R'}}_{a \succ b}| + 1$, which is symmetric to the earlier situation, so by *neutrality* a should win. Contradiction. ✓

Summary

We reviewed a large number of *voting rules* and observed:

- they explore different *intuitions* about how voting ‘should’ work
- they differ in view of the *profile information* they require
- they differ in view of the *normative principles* they satisfy
- they differ in view of their *computational requirements*

We finally saw an example for how to *characterise* a voting rule as the only rule that satisfies certain normative principles: *May’s Theorem*.

What next? More applications of the axiomatic method.