Homework #3

$$\boxed{\textbf{Deadline: Tuesday, 23 April 2024, 13:00}}$$

Submit your solutions for (up to) three of the following three exercises. If you solve all four, we will consult a random number generator to decide which three to look at and grade.

Exercise 1 (10 points)

Recall that a *constant-sum game* is a two-player normal-form game $\langle N, \boldsymbol{A}, \boldsymbol{u} \rangle$ for which there exists a constant $c \in \mathbb{R}$ such that $u_i(\boldsymbol{a}) + u_{-i}(\boldsymbol{a}) = c$ for all action profiles $\boldsymbol{a} \in \boldsymbol{A}$. Here, for either player $i \in N = \{1, 2\}$, we use $-i$ to refer to her opponent. In case $c = 0$, the game in question is a *zero-sum game*. For a given constant-sum game $\langle N, \boldsymbol{A}, \boldsymbol{u} \rangle$ with constant c, we can define a 'corresponding' zero-sum game $\langle N, \boldsymbol{A}, \boldsymbol{u}' \rangle$, i.e., a game with the same players and action profiles, for which $u_i'(\boldsymbol{a}) = u_i(\boldsymbol{a}) - \frac{c}{2}$ for all $i \in N$ and all $\boldsymbol{a} \in \boldsymbol{A}$. Intuitively, we would expect these two games to be 'similar' in many respects.

(a) The purpose of this first part of the exercise is to make the above intuition concrete as far as Nash equilibria are concerned. Show that the set of mixed Nash equilibria of a given constant-sum game does not change when we turn it into a zero-sum game by applying the transformation defined above.

(b) Give an example for why the strategies adopted by the players might change when we transform a constant-sum game as described above. This could be absolutely anything. You do not need to restrict yourself to concepts we have defined in class.

Exercise 2 (10 points)

We have seen that for zero-sum games fictitious play always converges: after a certain number of iterations, the profile of empirical mixed strategies will be (very close to) a Nash equilibrium. But we have said nothing about the *rate of convergence*. How long will this actually take? Write a program to simulate fictitious play for arbitrary 2×2 zero-sum games (and more general classes of games if you like). Then use your program to produce a (short) report that offers some insight into the question of the rate of convergence.

This exercise leaves you a lot of freedom for how to approach the problem and what questions to provide answers to exactly. Here are some of the issues you will have to engage with: How do you generate games to test your program on? How do you measure the 'distance' between two consecutive profiles of empirical mixed strategies, so as to be able to assert convergence? How do you best visualise the data you generate?

Exercise 3 (10 points)

Ignorance is bliss, they say. The objective of this exercise is to provide precise mathematical underpinnings for this intriguing piece of folklore wisdom.

To do so, construct a Bayesian game for which there is some uncertainty about which underling normal-form game is actually being played. This Bayesian game should have a Bayes-Nash equilibrium that provides strictly higher expected utility to all players than any of the Nash equilibria of any of the underlying normal-form games would. In other words, any way of resolving the uncertainty would leave everyone worse off.

Exercise 4 (10 points)

Consider the following variant of the family of congestion games. Everything works as in a regular congestion game, but each player has two possible types: they can be either *eager* or *indifferent*. The idea is the following. Each player must choose a set of resources to claim that would allow them to achieve their objective. An indifferent player's utility will depend only on the costs associated with those resources, while an eager player will derive some additional utility from being able to achieve their objective. More specifically, for an indifferent player we calculate her utility as for a regular congestion game, while for an eager player we add 100 points to that utility. Each player is eager with a prior probability of $\frac{1}{2}$.

(a) Provide a formal definition of this family of *Bayesian congestion games*, including in particular the utility functions of the players.

(b) Either prove of disprove the following claim:

   Every Bayesian congestion game has at least one pure Bayes-Nash equilibrium.

<u>Note:</u> The term 'Bayesian congestion game' has been used in a number of different senses in the literature. Consulting that literature probably will not be particularly helpful for solving this exercise. In any case, make sure you formalise and analyse the family of games indicated here, not some other family of games you found in the literature, and make sure you use the notation we use in the course, not some other notation you saw elsewhere.