Kenneth O. May: A set of independent necessary and sufficient conditions for simple majority decision

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1. May's results

2. From two alternatives to arbitrarily many: Arrow's critique

n individuals and two alternatives x and y

	Individual i		Group	
strictly prefers x to y	xP_iy	$D_i = 1$	xPy	$D = 1$
is indifferent to x and y	xI_iy	$D_i = 0$	xIy	$D = 0$
strictly prefers y to x	yP_ix	$D_i = -1$	yPx	$D = -1$

Group decision function: $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$

$$D = f(D_1, \ldots, D_n)$$

$N(1), N(0), N(-1)$: The number of D_i's with value 1/0/-1

Simple majority decision: The group decision function f defined by

$$f(D_1, \ldots, D_n) = \begin{cases} 1 & \text{if } N(1) > N(-1) \\ 0 & \text{if } N(1) = N(-1) \\ -1 & \text{if } N(1) < N(-1) \end{cases}$$

Condition I: The group decision function is decisive, i.e. defined and single valued for every element of $\{-1, 0, 1\}^n$.

Condition II: The group decision function is egalitarian.
$f(D_1, \ldots, D_n) = f(D_{j(1)}, \ldots, D_{j(n)})$ for all permutations j of $\{1, \ldots, n\}$

Condition III: The group decision function is neutral.
$f(-D_1, \ldots, -D_n) = -f(D_1, \ldots, D_n)$

Condition IV: The group decision function is positive responsive.
If $D_i' > D_i$ for some $i \in \{1, \ldots, n\}$
and $f(D_1, \ldots, D_{i-1}, D_i, D_{i+1}, \ldots, D_n) \in \{0, 1\}$
then $f(D_1, \ldots, D_{i-1}, D_i', D_{i+1}, \ldots, D_n) = 1$

Theorem

A group decision
is the simple majority decision
if and only if
it is always decisive, egalitarian, neutral, and positive responsive.

Proof.

"Only if": Quite trivial

"If": For given group decision function f satisfying conditions I-IV it is shown that

(1) $N(1) > N(-1)$ implies $D = 1$

(2) $N(1) = N(-1)$ implies $D = 0$

(3) $N(1) < N(-1)$ implies $D = -1$

using this consequence of condition II: D depends only on $N(1)$, $N(0)$, and $N(-1)$.

For reference:

(2) $N(1) = N(-1)$ implies $D = 0$
Condition III. Neutrality: $f(-D_1, \ldots, -D_n) = -f(D_1, \ldots, D_n)$
Condition I. Decisive

Proof of (2).

(2) is shown indirectly: Assume $N(1) = N(-1)$ and $D = 1$.
$f(D_1, \ldots, D_n) = 1$
By condition III: $f(-D_1, \ldots, -D_n) = -1$
But $(D_1, \ldots, D_n)$ contains the same number of 1's, 0's and -1's as $(-D_1, \ldots, -D_n)$
Contradiction (by condition I)
Similarly for assumption $N(1) = N(-1)$ and $D = -1$

For reference:

(1) $N(1) > N(-1)$ implies $D = 1$
(2) $N(1) = N(-1)$ implies $D = 0$
Cond. IV. Positive responsive: If $D_i' > D_i$ for some $i \in \{1, \ldots, n\}$
and $f(D_1, \ldots, D_{i-1}, D_i, D_{i+1}, \ldots, D_n) \in \{0, 1\}$
then $f(D_1, \ldots, D_{i-1}, D_i', D_{i+1}, \ldots, D_n) = 1$

Proof of (1).

(1) is shown by induction on $m = N(1) - N(-1)$
Induction start: Condition IV plus (2)
Induction step: Condition IV plus induction hypothesis

For reference:

(1) $N(1) > N(-1)$ implies $D = 1$
(3) $N(1) < N(-1)$ implies $D = -1$
Condition III. Neutrality: $f(-D_1, \ldots, -D_n) = -f(D_1, \ldots, D_n)$

Proof of (3).

(3) follows from Condition III plus (1)

✗ decisive
✓ egalitarian
✓ neutral
✓ positive responsive

$$f(D_1, \ldots, D_n) = \begin{cases} 1 & \text{if } N(1) \geq N(-1) \\ -1 & \text{if } N(1) \leq N(-1) \end{cases}$$

(majority decision where both alternatives are adobted when there is a tie)

✓ decisive
✗ egalitarian
✓ neutral
✓ positive responsive

$$f(D_1, \ldots, D_n) = \begin{cases} 1 & \text{if } D_1 + N(1) > N(-1) \\ 0 & \text{if } D_1 + N(1) = N(-1) \\ -1 & \text{if } D_1 + N(1) < N(-1) \end{cases}$$

(majority decision where the vote of individual 1 counts twice)

✓ decisive
✓ egalitarian
✗ neutral
✓ positive responsive

$$f(D_1, \ldots, D_n) = \begin{cases} 1 & \text{if } N(1) > 2 \cdot N(-1) \\ 0 & \text{if } N(1) = 2 \cdot N(-1) \\ -1 & \text{if } N(1) < 2 \cdot N(-1) \end{cases}$$

(two-thirds majority is needed for alternative x)

- ✓ decisive
- ✓ egalitarian
- ✓ neutral
- ✗ positive responsive

$$f(D_1, \ldots, D_n) = \begin{cases} 1 & \text{if } N(1) = n \\ 0 & \text{if } N(1) \neq n \neq N(-1) \\ -1 & \text{if } N(-1) = n \end{cases}$$

[Assuming that $n > 3$]

(unamity decision)

Kenneth O. May
A Note on the Complete Independence of the Conditions for Simple Majority Decision
Econometrica **21**, 172-73, 1953

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He remarks (*ibid.*, p. 680) that, "Since it follows that the pattern of group choice may be build up if we know the group preferences for each pair of alternatives, the problem [of determining group choices from the set] reduces to the case of two alternatives." This, however, would only be correct if transitivity were also assumed. Otherwise, there is no necessary connection between choices from two-member sets and choices from larger sets. If there are more than two alternatives, then it is easy to see that many methods of choice satisfy all of May's conditions, for example, both plurality voting and rank-order summation. A complete characterization of all social decision processes satisfying May's conditions when the number of alternatives is any finite number does not appear to be easy to achive.

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Group choice from $\{x, y\}$:	x
Group choice from $\{y, z\}$:	y
Group choice from $\{z, x\}$:	z
Group choice from $\{x, y, z\}$:	?

In Arrow's terms our theorem may be expressed by saying that any social welfare function (group decision function) that is not based on simple majority decision, i.e., does not decide between any pair of alternatives by majority vote, will either fail to give a definite result in some situation, favor one individual over another, favor one alternative over the other, or fail to respond positively to individual preferences.

In Arrow's terms our theorem may be expressed by saying that any social welfare function (group decision function) that is not based on simple majority decision, i.e., does not decide between any pair of alternatives by majority vote, will either fail to give a definite result in some situation, favor one individual over another, favor one alternative over the other, or fail to respond positively to individual preferences.

"social welfare function": Arrow's notion, arbitrarily many alternatives

"group decision function": May's notion, only defined for two alternatives

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Group decision function: $f : \{-1,0,1\}^n \to \{-1,0,1\}$

$$D = f(D_1, \ldots, D_n)$$

Condition II: The group decision function is egalitarian.
$f(D_1, \ldots, D_n) = f(D_{j(1)}, \ldots, D_{j(n)})$ for all permutations j of $\{1, \ldots, n\}$

Condition III: The group decision function is neutral.
$f(-D_1, \ldots, -D_n) = -f(D_1, \ldots, D_n)$

Condition IV: The group decision function is positive responsive.
If $D_i' > D_i$ for some $i \in \{1, \ldots, n\}$
and $f(D_1, \ldots, D_{i-1}, D_i, D_{i+1}, \ldots, D_n) \in \{0, 1\}$
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Simple majority decision: The group decision function f defined by

$$f(D_1, \ldots, D_n) = \begin{cases} 1 & \text{if } N(1) > N(-1) \\ 0 & \text{if } N(1) = N(-1) \\ -1 & \text{if } N(1) < N(-1) \end{cases}$$

R with indices and primes as defined by Arrow

S: The set of alternative social states

$\mathcal{R}$: The set of relations on S

$\widetilde{\mathcal{R}}$: The subset of $\mathcal{R}$ of relations that satisfy axiom I and II

$p : \mathcal{R} \to \mathcal{R}$ is a *relation permutation* on $\mathcal{R}$, if there is a permutation p' of S such that for all $R \in \mathcal{R}$,
$x \; p(R) \; y \leftrightarrow p'(x) \; R \; p'(y)$ for all $x, y \in S$

	x	y	z
x	1	1	1
y	0	1	1
z	0	0	1

	y	z	x
y	1	1	1
z	0	1	1
x	0	0	1

Social welfare function: $f : \widetilde{\mathcal{R}}^n \to \mathcal{R}$

$$R = f(R_1, \ldots, R_n)$$

Condition II: The social welfare function is egalitarian.
$f(R_1, \ldots, R_n) = f(R_{j(1)}, \ldots, R_{j(n)})$ for all permutations j of $\{1, \ldots, n\}$

Condition III: The social welfare function is neutral.
$f(p(R_1), \ldots, p(R_n)) = p(f(R_1, \ldots, R_n))$ for all relation permutations p on $\mathcal{R}$

Condition IV: The social welfare function is positive responsive.
If (xR_iy and not $yR_i'x$) or (not xR_iy and $xR_i'y$) for some $i \in \{1, \ldots, n\}$ and $R_j = R_j'$ for all $j \neq i$ and xRy then not $yR'x$

Simple majority decision: The social welfare function such that for all alternatives x and y, xRy iff the number of individuals such that xR_iy is at least as great as the number of individuals such that yR_ix.

The voting function of individual i, V_i: A function from S to $\mathbb{R}$ (with possible restriction)

$\mathcal{V}$: The set of possible voting functions.

0: Tie (assumed not in S)

Given a permutations p of $S \cup \{0\}$ such that $p(0) = 0$, the associated *function permutation* $\bar{p} : \mathcal{V} \to \mathcal{V}$ is defined by $\bar{p}(V)(x) = V(p(x))$ for all $V \in \mathcal{V}$ and $x \in S$

Election function: $f : \mathcal{V}^n \to S \cup \{0\}$

Condition II: The election function is egalitarian.
$f(V_1, \ldots, V_n) = f(V_{j(1)}, \ldots, V_{j(n)})$ for all permutations j of $\{1, \ldots, n\}$

Condition III: The election function is neutral.
$f(\bar{p}(V_1), \ldots, \bar{p}(V_n)) = p(f(V_1, \ldots, V_n))$ for all permutations p of $S \cup \{0\}$ such that $p(0) = 0$

Condition IV: The election function is positive responsive.
If $V_i'(x) > V_i(x)$ for some $i \in \{1, \ldots, n\}$ and some $x \in S$
and $V_i'(y) \leq V_i(y)$ for all $y \neq x$
and $V_j' = V_j$ for all $j \neq i$
and $f(V_1, \ldots, V_{i-1}, V_i, V_{i+1}, \ldots, V_n) \in \{0, x\}$
then $f(V_1, \ldots, V_{i-1}, V_i', V_{i+1}, \ldots, V_n) = x$

Simple majority decision: The voting functions V_i are all such that there is at most one $x \in S$ such that $V_i(x) = 1$ and for all other $y \in S$, $V_i(y) = 0$. The election function is such that for all $x \in S$, $f(V_1, \ldots, V_n) = x$ iff $V_1(x) + \cdots + V_n(x) > V_1(y) + \cdots + V_n(y)$ for all $y \neq x$, and $f(V_1, \ldots, V_n) = 0$ no element of S satisfies this condition.

Rank-order election: The voting functions V_i are all with codomain $\{0, 1, \ldots, m\}$, where m is the number of alternatives in S, and such that each of $1, \ldots, m$ is the value of at most one element of S. The election function is (again) such that for all $x \in S$, $f(V_1, \ldots, V_n) = x$ iff $V_1(x) + \cdots + V_n(x) > V_1(y) + \cdots + V_n(y)$ for all $y \neq x$.

Also satisfies conditions II-IV.

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Kenneth J. Arrow
Social Choice and Individual Salues
Second edition, John Wiley & Sons, Inc., New York, 1963