Voting in Parallel Universes

ILLC Workshop on Collective Decision Making 2015

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Voting

- There is no perfect voting rule
- There is no consensus on using a particular rule
- Ties do occur
- Some voting rules tend to have a large set of winners.

Can we use existing rules to define rules that are more decisive and less sensitive to tie-breaking rules?

Notations

- N is the set of n **voters**
- C is the set of m **candidates**
- each voter has a preference $\succ_i$ over the set of candidates.
  We assume the preference is a *linear order* over the set of candidates
  We can also view a *linear order* as a *permutation*.
  So we will write $\mathcal{S}(X)$ the set of all permutations/linear orders on the set X.
- a **profile** is an element of $\mathcal{S}(C)^n$, i.e. a vector $\langle \succ_1, \ldots, \succ_n \rangle$

Definition ((Irresolute) Social Choice Function)

A social choice function is a mapping $f : \mathcal{S}(C)^n \to 2^C$

The set $f(\langle \succ_1, \ldots, \succ_n \rangle)$ is the set of **winners**.

Dealing with Ties

- breaking ties by breaking *anonymity*: we break a tie using the preference of a special *voter* (e.g. the president of a committee breaks the ties, or the oldest)
  ⇨ not all voters are equal
- breaking ties by breaking *neutrality*: we break a tie using some relation over the *candidates*: break the ties in favor of the oldest/yougest candidate or using lexicographic order on their names
  ⇨ not all candidates are equal

We will focus on the approach breaking neutrality.

Definition (Permutation rule)

We call a *permutation rule* a mapping

$$f : \mathcal{S}(C)^n \times \mathcal{S}(C) \to C$$

We can view f as an irresolute voting rule attached with a tie-breaking rule $\triangleright$.

new rules: each tie-breaking defines a different universe

Given a *permutation rule* f, we can define two new irresolute voting rules:

union rule $uf : \mathcal{S}(C)^n \to 2^C$ such that

$$uf(\langle \succ_1, \ldots, \succ_n \rangle) = \{c \in C \mid \exists \rhd \in \mathcal{S}(C) \mid f(\langle \succ_1, \ldots, \succ_n \rangle, \rhd) = c\}$$

This rule selects the candidates that win at least once with a permutation rule.

argmax rule $af : \mathcal{S}(C)^n \to 2^C$ such that

$$af(\langle \succ_1, \ldots, \succ_n \rangle) = \max_{c \in C} |\{\rhd \in \mathcal{S}(C) \mid f(\langle \succ_1, \ldots, \succ_n \rangle, \rhd) = c\}|$$

This rule selects the candidates that most often win over all permutation rules.

a new social decision scheme

A Social Decision scheme if a mapping

$$\mathcal{S}(C) \to \Delta(C)$$

where $\Delta(C)$ denotes the set of all *probability distributions* over the set of candidates.

frequency rule Given a *permutation rule* f, we can define a new social decision scheme $pf : \mathcal{S}(C)^n \to 2^C$ such that

$$pf(\langle \succ_1, \ldots, \succ_n \rangle)(c) = \frac{|\{\rhd \in \mathcal{S}(C) \mid f(\langle \succ_1, \ldots, \succ_n \rangle, \rhd) = c\}|}{n!}$$

Case study: Single Transferable Vote (also called Instant Run-Off Voting)

STV is an iterative rule that works as follows:

- at each round, each voter casts a ballot containing its favourite candidate
- We cound the number of votes for each candidate
  - if a voter obtains a majority: it is elected
  - otherwise we **eliminate** the candidate with the **smallest** number of votes and we iterate the process with the reduced set of candidates
- the process eventually stops as either a candidates gets the majority or because it is the only candidate left

➩ there can be ties between candidates that got the smallest number of votes!

Example

- 10 voters named 1, 2, …, 10
- 3 candidates a, b and c
- we note the preference $a \succ b \succ c$ as abc

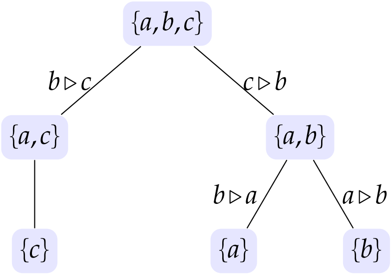
number of voters	preference
4	$a\ b\ c$
3	$b\ c\ a$
2	$c\ b\ a$
1	$c\ a\ b$

a gets 4 votes, b and c are tied with 3
➩ two universes: one where b is eliminated the other where c is eliminated

- when b is removed: c wins
- when c is removed: a and b are tied again!

Conitzer, Ronglie and Xia (IJCAI-09) called this STV with parallel universes.

Tree representation



- For each leaf nodes, we must count the number of tie-breaking rules that satisfy the "constraints"
- ⇨ "counting the linear extensions" and it is a #-P complete problem.
- when ties are *always* between only *two* candidates, we can count in polynomial time.

sketch of proof

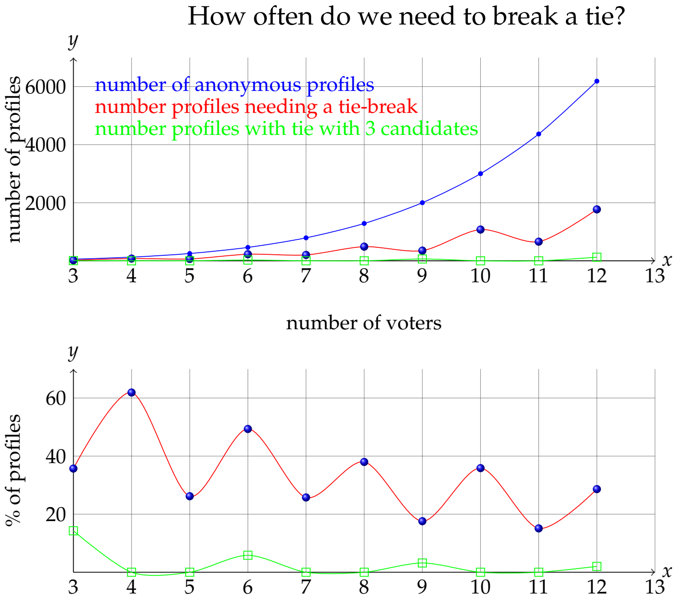
- There is a tie between two candidates a and b at node V
- There are three types of constraints:
  - $x_i \triangleright a$ (it cannot be $a \triangleright x_i$ as otherwise a would have been eliminated); let assume there are k such constraints
  - $y_j \triangleright b$; let us assume l such constraints
- constraints that contains neither a nor b
- note that we cannot have $x_i = y_j$ for all $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$
- we cannot have a constraint that include a x_i and a y_j (e.g. $x_i \triangleright y_j$ or $y_j \triangleright x_i$ for all $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$)

⇨ For the branch corresponding to the constraint $a \triangleright b$: count the number of sequences of length $k+l+2$ for interspersing $x_1 x_2 \ldots x_k a$ with $y_1 y_2 \ldots y_l b$ such that a is before b.
⇨ choose the position of $k+1$ elements (corresponding to the x_i and a) among the $k+l+1$ possible positions
⇨ choose $k+1$ elements from a set of $k+l+1$ elements.

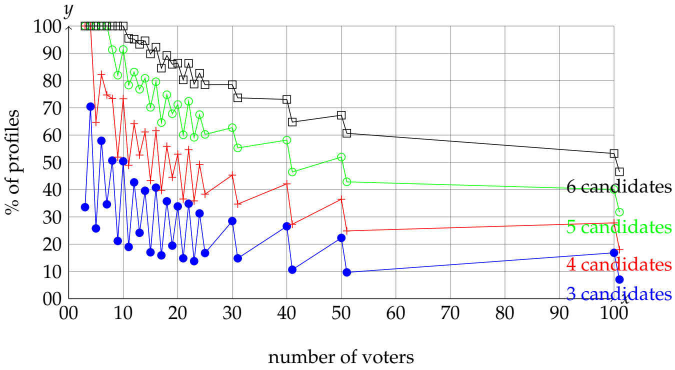
case with ties between more than 2 candidates

- Assume a tie between three candidates, say a, b and c.
- say we eliminate a, the constraints down this branch are $a \triangleright b$ and $a \triangleright c$.
- the following constraints are feasible:
  - $x_i \triangleright a$
  - $y_j \triangleright b$
  - $z_k \triangleright c$
- It is now possible to have a constraint $x_i y_j$ as there could have been a tie between x_i, y_i and a in which x_i is eliminated.
- our combinatorics argument will not work in this case.

3 candidates Anonynous Culture



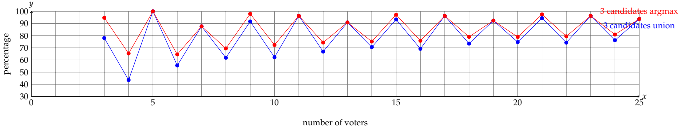
Sampling Impartial Culture



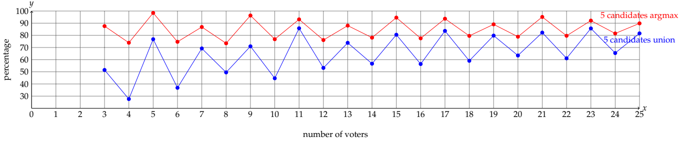
sampling 100,000 profiles with impartial culture
number of times a tie-breaking rule is needed

Decisiveness – Sampling Impartial Culture

3 candidates



5 candidates



sampling 10,000 profiles with impartial there is a unique winner

Future Work

- Banks: there is a polynomial algorithm to get one Banks winner
- ⇨ union rule provides all Banks winner
- ⇨ argmax rule discriminates among all Banks winner
- Complexity: we conjecture #-P complete for STV
- axiomatic: if the voting rule has some properties, what is conserved by union and argmax. Immediate for some axioms, not clear for others.
- Top Cycle tends to have a large winner set. Does the argmax rule helps to improve decisiveness?
- Comparison with perturbation method (Freeman, Conitzer, Brill AAMAS-15)
- Social Decision Scheme: we can propose particular SDS using our frequency rule. What are the properties of such rules?