Local Envy-Freeness in House Allocation Problems

Workshop on Collective Decision Making - Amsterdam

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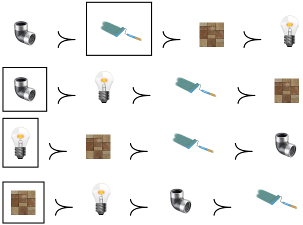
House allocation

- n agents
- n indivisible and unsharable resources/objects
- Each agent has *strict ordinal preferences* over the objects

⇒ Each agent must receive exactly one resource

Example: 4 workers, 4 tasks: wall painting  , tile laying  , plumbing  and electricity 

Alice:
Bob:
Carlos:
Diana:



Envy-freeness in house allocation

Envy-freeness: No agent prefers the object allocated to another agent to her assigned resource

4 workers: Alice and Bob (morning), Carlos and Diana (afternoon)

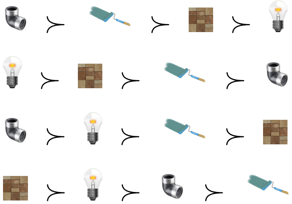
4 tasks: wall painting  , tile laying  , plumbing  , electricity 

Alice:

Bob:

Carlos:

Diana:



$\Rightarrow$ **No envy-free allocation**

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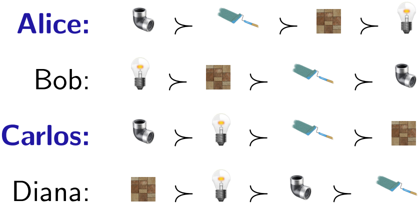

However, Alice and Carlos never meet

$\Rightarrow$**Envy?**

Envy-freeness in house allocation

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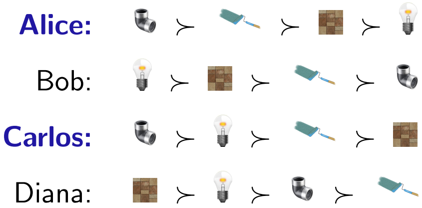
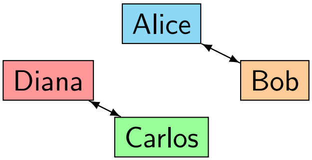


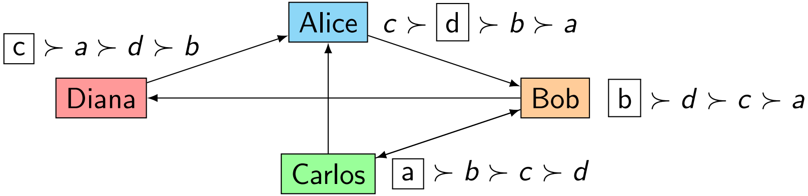


$\Rightarrow$ **Local envy-freeness?**

Local Envy-Freeness (LEF)

- Social network: captures the possibility of envy among agents
  - Represented by a *directed graph* over the agents
  - Local envy: an agent envies an agent who is "visible" for her, i.e. who is a successor in the graph

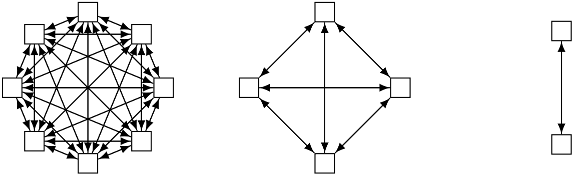
⇒ Locally envy-free (LEF) allocation:
  - No agent prefers the object allocated to a successor agent in the graph to her assigned resource



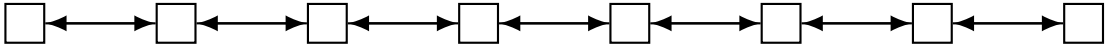
⇒ Not envy-free but locally envy-free

Meaningful structures of graphs

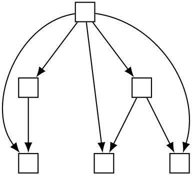
- partition: disjoint groups of agents → cluster graphs



- a line (time schedule)



- hierarchical structures



Issues

Centralized approach:

- Is it possible to construct an LEF allocation?
- Is it possible to place the agents on a given graph and to assign them resources such that the allocation is LEF?

Distributed approach:

- Are the agents able to reach an LEF allocation by exchanging their objects?

Outline

Existence of an LEF allocation

Resource and location allocation

Reachability of an LEF allocation

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Existence of an LEF allocation

Resource and location allocation

Reachability of an LEF allocation

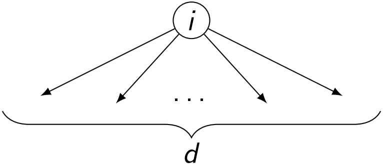
Condition for an allocation to be LEF

If object x is assigned to agent i, then all the successors of i must get objects that i prefers less than x

$\Rightarrow$ The best object of i must be assigned either to i or to an agent not visible for i

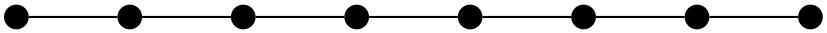
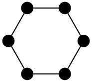
$\Rightarrow$ Each agent i with d successors must get an object which is ranked among her $n - d$ best objects

$[x_1 \succ x_2 \succ \cdots \succ x_{n-d}] \succ$ ~~$x_{n-d+1} \succ \cdots \succ x_{n-1} \succ x_n$~~



Complexity results w.r.t. the structure of the graph

Does there exist a locally envy-free allocation?

- NP-complete even for an undirected graph which is
  - a matching 
  - a line 
  - a circle 
  - a cluster graph (set of disjoint cliques)
- Solvable in polynomial time when the graph is
  - a Directed Acyclic Graph (DAG)

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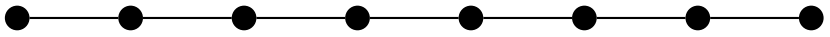
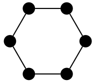
  - **a line** 
  - **a circle** 
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From the matching to the line and the circle

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$x_1 \succ x_2 \succ \cdots \succ x_n \succ d$

$d \succ x_1 \succ x_2 \succ \cdots \succ x_n$

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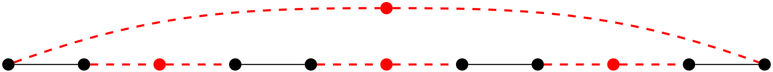

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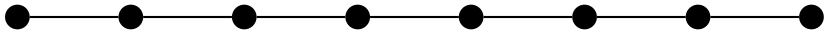
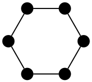
  - a line 
  - a circle 
  - a **cluster graph** (set of disjoint cliques)
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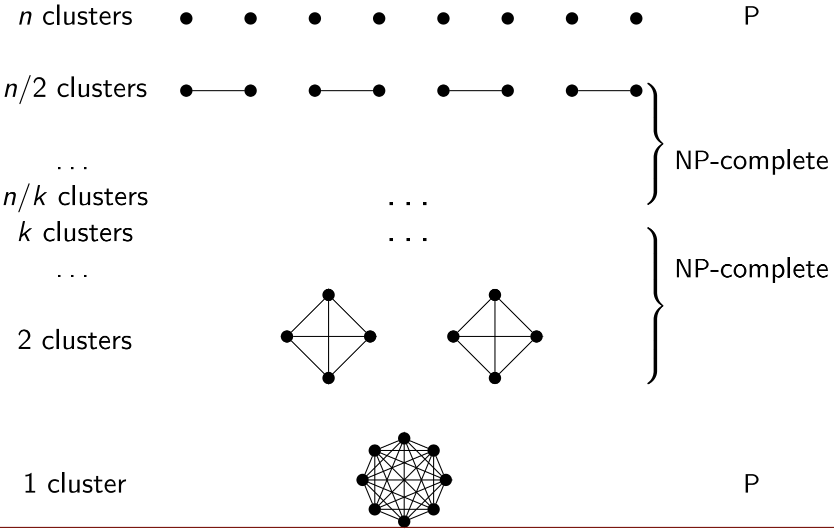
Cluster graphs



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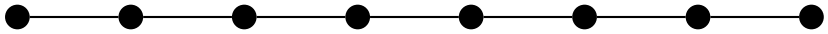
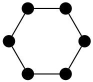
  - a line 
  - a circle 
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- Solvable in polynomial time when the graph is
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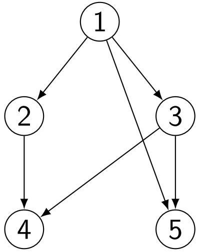
Directed Acyclic Graphs

An LEF allocation always exists and can be found in polynomial time

⇒ Serial Dictatorship from the "source" agents



1 :	a	$\succ$	b	$\succ$	c	$\succ$	d	$\succ$	e
2 :	a	$\succ$	b	$\succ$	e	$\succ$	c	$\succ$	d
3 :	b	$\succ$	d	$\succ$	c	$\succ$	e	$\succ$	a
4 :	d	$\succ$	c	$\succ$	a	$\succ$	b	$\succ$	e
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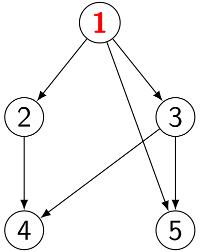

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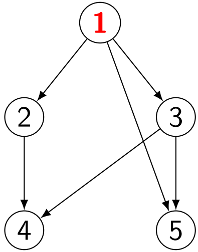

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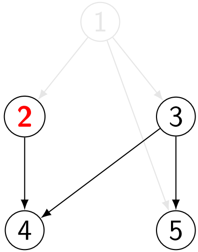

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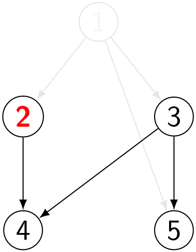

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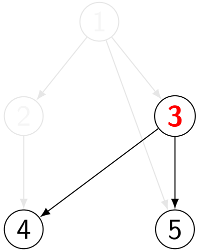

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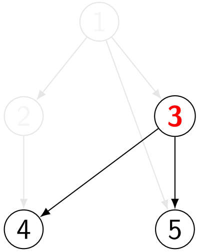

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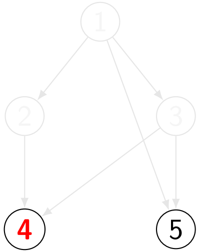

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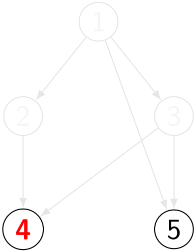

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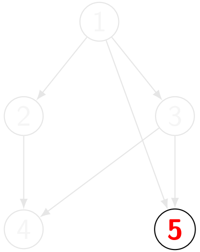

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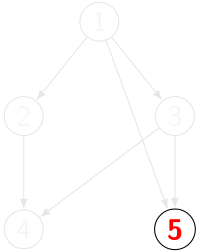

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Complexity results w.r.t. the out-degree of the graph

Does there exist a locally envy-free allocation?

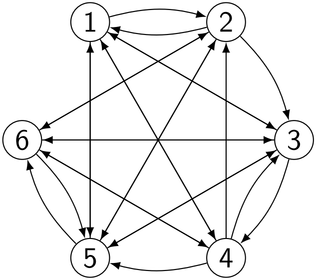
δ^+: out-degree of the graph

- NP-complete even for:
  - out-degree $\delta^+ \leq k$ (for a constant $k \geq 1$)
  - out-degree $\delta^+ \geq n - k$ (for a constant $k \geq 3$)

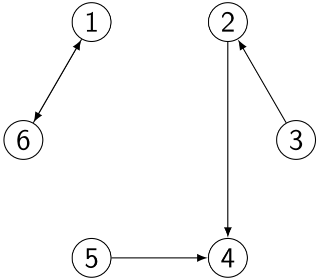
- Solvable in polynomial time when $\delta^+ \geq n - 2$ ("very dense graphs")

Very dense graphs

Agents connected to any other agent or to anyone except one



Non-envy graph:
out-degree ≤ 1



For each best object of an agent:
- allocate it to her **OR**
- allocate it to her non-successor agent

$\Rightarrow$ 2-SAT formula

Outline

Existence of an LEF allocation

Resource and location allocation

Reachability of an LEF allocation

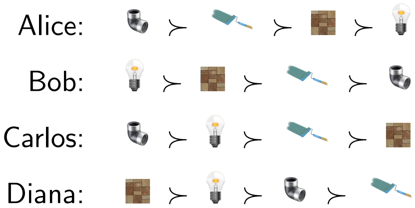
Motivation

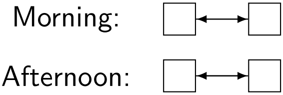
What if the central authority has also the power to assign the agents to positions in the graph?

⇒ Typical example of tasks and time schedule

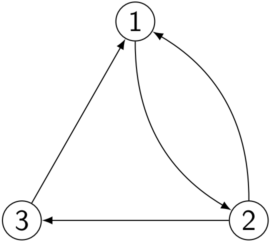
4 workers: Alice, Bob. Carlos and Diana

4 tasks: wall painting, tile laying, plumbing, electricity
2 time slots: 2 workers on the morning, 2 workers on the afternoon





Allocating both the objects and the locations: example



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$\Rightarrow$ No locally envy-free allocation

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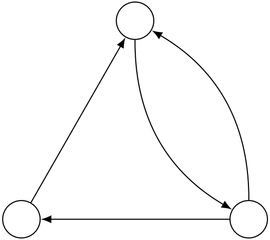

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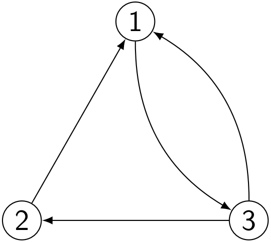

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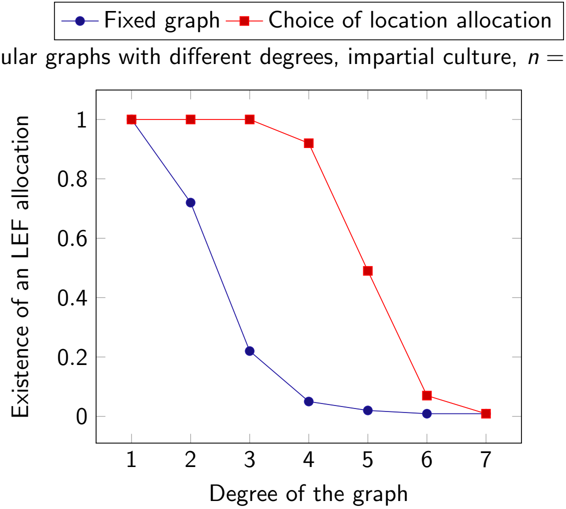
$$

Locally envy-free allocation!

Likelihood of finding an LEF allocation



Complexity results

Is it possible to find an allocation of the agents to the vertices of the graph and an allocation of objects to the agents such that we get an LEF allocation?

- NP-complete for a general graph, even undirected
  - Reduction from Independent Set
- Solvable in polynomial time for very dense undirected graphs

Polynomial algorithm for very dense undirected graphs

Graph G: degree $\geq n-2 \rightarrow$ Non-envy graph $\overline{G}$ = matching

1. Construct the non-envy graph
   - Each agent must be coupled with the only other agent with the same top object
     $\rightarrow$ if not possible: return **false**
2. Apply the 2-SAT algorithm for finding an LEF allocation in G

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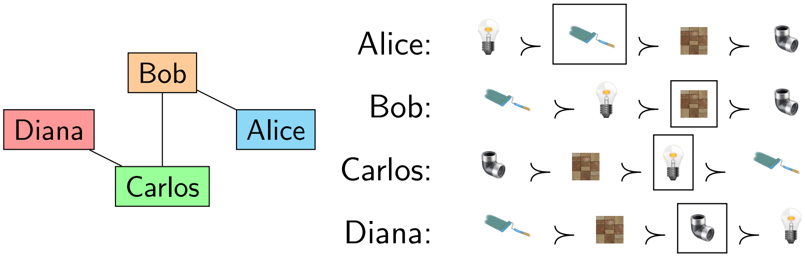
Reachability of an LEF allocation

Setting: housing market with social constraints

- Initial allocation
- Exchanges among agents conditioned by the social network
  - Social network: *undirected graph*
- Rational swaps:
  - Swaps involving two neighbors in the graph
  - The agents must be better off after each exchange



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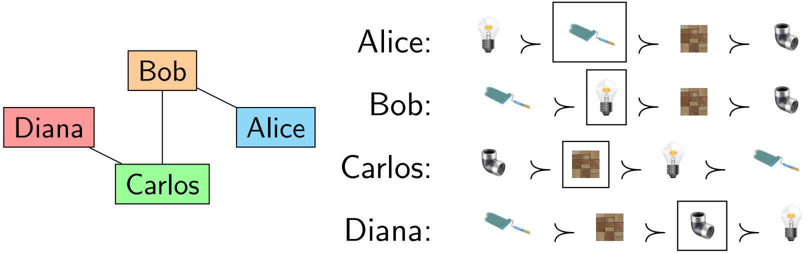
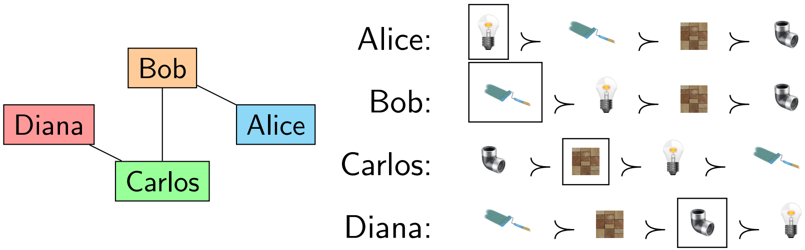


Sequence of exchanges: $(\{\mathit{Bob}, \mathit{Carlos}\}$

Setting: housing market with social constraints

- Initial allocation
- Exchanges among agents conditioned by the social network
  - Social network: *undirected graph*
- Rational swaps:
  - Swaps involving two neighbors in the graph
  - The agents must be better off after each exchange



Sequence of exchanges: $(\{Bob, Carlos\}, \{Alice, Bob\})$

Setting: housing market with social constraints

- Initial allocation
- Exchanges among agents conditioned by the social network
  - Social network: *undirected graph*
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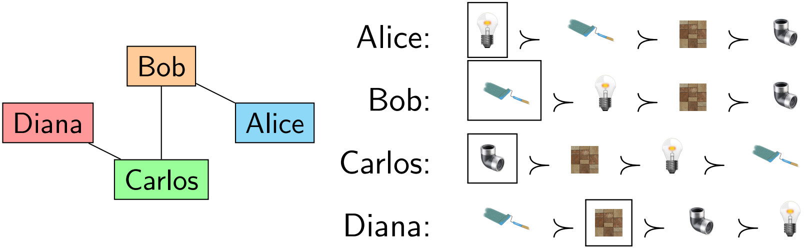


Sequence of exchanges: $(\{Bob, Carlos\}, \{Alice, Bob\}, \{Carlos, Diana\})$

Reachability and local envy-freeness

- Reachability: An allocation is reachable if there exists a sequence of rational swaps from the initial allocation leading to this allocation
- Stability: An allocation is stable if no rational swap is possible

$\Rightarrow$ An LEF allocation is necessarily stable

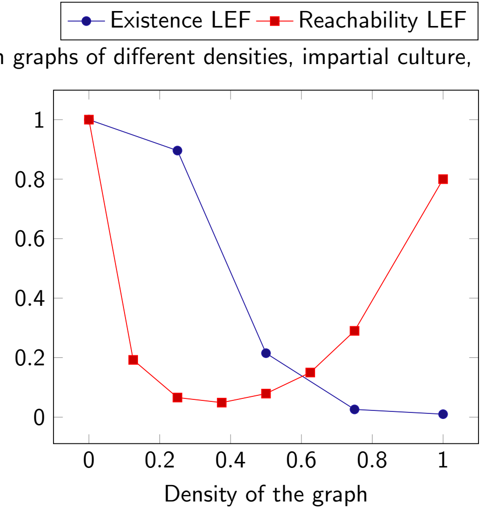
Does there exist a sequence of rational swaps from the initial allocation leading to an LEF allocation?

Complexity results

Does there exist a sequence of rational swaps fron the initial allocation leading to an LEF allocation?

- NP-complete even when the social network is
  - a tree
  - a complete graph
- Solvable in polynomial time when the social network is
  - a matching
  - a star

Likelihood to reach an LEF allocation



Outline

Existence of an LEF allocation

Resource and location allocation

Reachability of an LEF allocation

Conclusion

Summary

- Relaxing the strong requirement of envy-freeness in house allocation
  - Local envy-freeness defined according to a graph structure
  - Enables to take into account more realistic situations
- Locally envy-free allocations more likely to exist in sparse graphs
- Good approximation algorithms (not presented in this talk)
  - Minimization of the number of locally envious agents
  - Minimization of the average degree of envy
- Computationally:
  - raises many interesting questions
  - usually hard to solve for many graphs, even very simple ones
  - but some interesting cases can be solved efficiently
    - very dense undirected graphs
    - directed acyclic graphs

Perspectives

- Extension to multiple resources per agent
  - Several recent papers deal with a local envy-freeness notion in this framework
- Domaine restriction?
- Initial partial allocation to complete?
- Relation with Pareto-efficiency?