1 Introduction

It may be taken for granted that any attempt at defining disorder in a formal way will lead to a contradiction. This does not mean that the notion of disorder is contradictory. It is so, however, as soon as I try to formalize it.

Hans Freudenthal

0111000101001000....... is an initial segment of a long sequence produced by tossing a coin. In its broadest outline, the subject of this thesis is the mathematical description of the sequences produced by random processes (such as coin tossing), which will be called *random sequences*. The tantalizing motto, taken from Freudenthal [29], expresses a negative verdict on this enterprise. But meanwhile it raises a no less interesting question: How can a non-contradictory concept *necessarily* defy formalisation?

Clearly, the two definite articles in the phrase "the mathematical description of the sequences produced by random processes" present a host of problems. There might not be such a description, as Freudenthal thinks; or it might be completely trivial, the reason being that all we can say a priori on the sequences produced by, say, coin tossing is, that these are sequences of zeros and ones. On the other hand, there *do* exist various definitions of random sequences; perhaps even too many.

The discussion in the pages that follow is therefore concentrated on two main questions:

1. Is a mathematical definition of random sequences possible and if so, why should one want to give such a definition?
2. Given the fact that various definitions have been proposed, does it make sense to ask for criteria which allow us to choose between them?

Even apart from its usefulness, the possibility of providing a definition of randomness has often been doubted. Here is a grab-bag of some of the a priori reasons which have been adduced for this conviction:

– As soon as you can define randomness, it ceases to be true randomness;
– Randomness is a property of processes, not of the sequences generated by such processes;
– It is characteristic of a random process that it may generate *any* sequence.

For the moment, we shall leave these a priori arguments unanalyzed.

The first person to discuss systematically the possibility, and indeed the necessity, of a definition of random sequences was Richard von Mises, who provided an axiomatisation of probability theory with "random sequence" as a primitive term. He argued, as did later

Kolmogorov, that, if probability is interpreted as relative frequency, then the applicability of probability theory to real phenomena (which is amply verified) entails that these phenomena must have certain properties of randomness, and he proposed to take these properties as basic for a definition of random sequences (other properties being optional).

A definition of randomness is therefore necessary to explain the applicability of probability theory and a minimal set of properties random sequences have to satisfy can be deduced from its rules, a priori reasons for the impossibility of such a definition notwithstanding. But in a philosophical analysis we must of course investigate the apparent conflict between compelling physical reasons *pro* and a priori reasons *contra* a mathematical definition of randomness.

Although in the thirties, but also in recent years, there has been a lively commerce in definitions of randomness, our second question, namely: Do there exist criteria to choose between these definitions?, has not been explicitly discussed in the literature. To be sure, there have been discussions among partisans of various schools, the Geneva conference on the foundations of probability (1937) being a notable example. But one is struck by the sheer monotony of these discussions, the same arguments pro and con being repeated over and over again, without noticible effects upon the opinions of the discussants.

We propose to break this stalemate by analyzing possible sources for the lack of mutual comprehension so clearly displayed. The conclusion of our analysis will be that von Mises, around whose axiomatisation of probability theory the discussion centered, had views on the foundations of probability and of mathematics in general, which were not shared, but also not fully understood, by his critics. His view on the foundations of mathematics, usually expressed only implicitly, was shaped by his work as an applied mathematician and is a mixture of constructivism and a tendency to introduce bold concepts whenever the description of real phenomena seems to necessitate it. From this mixture results what one might call an inhomogeneous mathematical universe, which is a far cry from the very homogeneous set theoretical universe that inspired some of the objections of his critics. Unfortunately, these assumptions were not made explicit in the debate. Von Mises' views on the foundations of probability did, of course, figure explicitly and prominently in the debate; but its critics did not show a reciprocal awareness of their own assumptions, thereby successfully creating the impression that, while von Mises' view was unnecessarily complicated, theirs was simplicity itself. We believe that this debate, if analyzed correctly, points to the conclusion that different (objective) interpretations of probability lead to different requirements for random sequences. Hopefully, this point of view is helpful in understanding the debate that raged between von Mises and his critics; and if it directs the reader away from bickering about definitions of randomness and toward the deeper questions of the foundations of probability, it has fulfilled its purpose.

These two main questions determine much of the technical work in Chapters 3 to 5. Given that there exist different types of definitions, each with its own minor variants, we must investigate how these definitions are related extensionally. Some of these relationships are well know, e.g. that between randomness in the sense of Martin-Löf and definitions involving (variants of) *Kolmogorov complexity*. In these cases, the novelty of our treatment consists solely in introducing new proof techniques. But other relations have been studied less thoroughly, notably that between von Mises' semi-formal definition, based on so called *admissible place selections* , and the other types. There are obvious reasons for this lack of attention. The fact that von Mises' definition is not quite formal renders a comparison with the other definitions, which *are* rigorous in the modern sense, difficult; and often the need for such a comparison is not felt acutely because, say, Martin-Löf's definition is considered to be an *improvement* on von Mises' proposal, rather than an *alternative*, based on radically different principles. We therefore have to study ways in which to make von Mises' definition precise; moreover, these attempts to instill precision should be based as much as possible upon his own philosophical premises. This problem has not been solved entirely, but the results that have been obtained, do enable us to effect a rigorous comparison between von Mises' definition and the other types.

So far, we have been concerned with *extensional* relationships between different *types* of definitions. But the exact meaning and justification of the definitions is no less important. The rationale behind von Mises' definition is studied extensively in Chapter 2. It turns out that it is completely justified on von Mises' own interpretation of probability, to be called *strict frequentism* . His opponents, on the other hand, usually start from a very different interpretation of probability, the *propensity interpretation*, and it is this interpretation which inspires those modern definitions of randomness which, following Martin-Löf, characterise randomness as the satisfaction of certain statistical tests. The choice of the particular type of test employed by Martin-Löf is, in our opinion, debatable, and the conclusion of the discussion is, in a nutshell, that whereas von Mises' definition is philosophically rigorous, but technically less so, with Martin-Löf's definition it is just the other way around.

The most promising approach to the characterisation of randomness appears to be the one inaugurated by Kolmogorov, using a notion of *complexity* for finite sequences. Philosophically, it stands midway between the definitions of von Mises and of Martin-Löf. Kolmogorov accepts von Mises' view that an analysis of the conditions of applicability of probability theory necessarily leads to the concept of a random sequence (although they differ in the place they allot to random sequences in the formal structure of probability theory). But while von Mises requires of random sequences only those properties which make the deductions of probability theory go through, Kolmogorov goes further and in a sense explains,

non-probabilistically, why random sequences must have those properties.
Technically, a major advantage of Kolmogorov complexity is, that it allows us to discuss degrees of randomness, both of sequences as a whole and within a single sequence. This feature leads to new problem which cannot be posed in the other frameworks, such as: What is the connection (if any) between the arithmetical complexity and the Kolmogorov complexity of a sequence? What is the connection between traditional measures of disorder such as entropy in its various forms, and Kolmogorov complexity?

The structure of this book is as follows. It consists of two parts, the first historical, the second technical, which can by and large be read independently. Each of the technical Chapters (3 to 5) has its own non-technical introduction, setting out the reasons for the constructions that follow. Of course, for the full motivation of the technical work, the reader is referred to the historical part.
Chapter 2 deals with von Mises' semi-formal definition of random sequences and its function in his axiomatisation of probability theory. We believe that insufficient attention to this context has lead to wholly unjustified criticisms of von Mises' theory. Accordingly, an overview of, and a critical commentary on, the debate to which von Mises' introduction of random sequences gave rise, will occupy half of the chapter. As intimated already, one of the main conclusions of our analysis will be that the evident lack of mutual understanding displayed in this debate is a consequence of widely divergent interpretations of probability.

The second, technical, part of the thesis opens with a chapter on Martin-Löf's definition of random sequences via statistical tests, and some of its variants. The emphasis of the discussion is, technically, on unified methods of proof, and, philosophically, on the virtues and vices of the particular type of test adopted by Martin-Löf. The fourth chapter is an updated version of [53] and contains results relating the definition of randomness of von Mises and that of Martin-Löf and its variants. In particular we study the behaviour of Martin-Löf random sequences under so called place selections (theorem 4.5.2) and we give a new proof of a famous theorem due to Ville, the philosophical implication of which is discussed in 2. We hope that this proof (4.6.1) has more explanatory power than Ville's original combinatorial argument.
Chapter 5 is concerned with what we consider to be the most promising development incited by von Mises' original proposal: Kolmogorov complexity. After studying several of its variants, we settle for the definition proposed by Chaitin, which allows an equivalent condition for randomness in the sense of Martin-Löf (5.4.3). As mentioned above, the decisive advantage of a complexity theoretic definition vis à vis other definitions of randomness is that it also enables us to measure the degree of randomness of a sequence, both locally, as a function of the initial segments of the sequence, and globally, as a number

attached to the sequence as a whole. The local behaviour of the degree of randomness is studied in a section on complexity oscillations (5.4, especially 5.4.2-3); the global behaviour is compared with measures of disorder defined in ergodic theory, such as metric entropy (5.5.2) and topological entropy (5.5.3).
The appendix (Chapter 6) contains notations and definitions not explained in the text.

Passages detracting from the main argument are labelled **Digression**. Those who are interested more in philosophical vistas than in technical details will find at the beginning of each chapter a list of sections which do not bear directly on foundational issues. But some mathematics is necessary; there is no royal road to the philosophy of science.