EXTENSION PROBLEMS IN INTUITIONISTIC PLANE PROJECTIVE GEOMETRY

D. VAN DALEN

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ACADEMISCH PROEFSCHRIFT

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DIRK VAN DALEN

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DRUKKERIJ HOLLAND N.V. — AMSTERDAM

Promotor: Prof. Dr. A. Heyting

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Harry

Introduction

On several occasions Professor HEYTING has commented on the axiomatic method in intuitionistic mathematics [5], [6, p. 50], [8]. Before proceeding to our actual subject, we shall repeat some of his remarks on the intuitionistic view on axiomatics. In intuitionistic mathematics we deal with objects which have been constructed. For this reason the axiomatic method cannot be understood in its *creative* function [8]. In its *descriptive* function it is applicable both in intuitionistic mathematics and in classical mathematics. However, we need not limit ourselves so far as to consider the axiomatic method in its descriptive function alone. We can apply the axiomatic method even when it is unknown whether there exists a model, then we attach the following meaning to it:

Suppose we derive in the intuitionistic sense from a set of axioms (assumptions) A a theorem T, then whenever we construct a set S of objects, satisfying A, we know that T holds for S.

As the construction of models is exceedingly sophisticated, there are many open problems in this field.

We are dealing here with several extension problems in intuitionistic geometry. The first problem is the extension of an affine plane in a natural way to a projective plane. Professor HEYTING investigated this problem for the first time [7]. He added three more axioms to his list of axioms for the affine plane in order to accomplish the demanded extension. Here we prove the dependence of two of them on the axioms for the affine plane. It is still unknown whether the remaining axiom is dependent on or independent of the axioms for the affine plane, only partial results were achieved.

The actual extension, i.e. the construction of new points and lines, does not raise insurmountable difficulties. The real difficulty is met with in proving the axioms for the projective plane. This is not surprising, considering the existential quantifier in P_1 and P_2.

Though we cannot affirm the existence of a projective extension of an affine plane, we at least know (theorem 2) that if an affine plane possesses a projective extension, this extension is determined up to an isomorphism.

The divergence of this theory from the classical theory is mainly caused by the fact that where the classical theory considers only two kinds of points (proper and improper), we also must introduce points of which it is unknown whether they are improper or proper.

An analogous problem provides the theory of coordinatization. It is well known that in the classical theory a ternary field determines an affine plane (and even a projective plane) up to an isomorphism. Only a weaker version is proved here: If a ternary field determines an affine plane, then the affine plane is determined up to an isomorphism. The last problem to be considered is the extension of the pseudo-ordering of a ternary field T to a cyclical ordering of a projective plane, determined by T. Miss CRAMPE solved this problem in the classical theory [1]. The extension is accomplished here intuitionistically, always assuming that the ternary ring determines a projective plane.

A number of classical extension problems are not represented here. Especially those, that are concerned with incidence-structures and free extensions. Indeed, the notion of an incidence structure seems too difficult for a general intuitionistic treatment. An incidence structure with apartness relation appears to be rather unmanageable. To obtain a significant notion of incidence structure one will have to suppose by definition its imbedding in a projective plane. On the other hand, this would deprive it of much of its character. Of course, there is no objection against incidence structures in which both points and lines form discrete species. These reflections show that incidence structures are not especially helpful for the intuitionist in constructing models.

The axiom systems for the projective plane and the affine plane were taken from [7]. The axiom system for the ternary field was developed by Professor HEYTING in his lectures during the course 1956–1957. We have omitted proofs when they did not differ essentially from the proofs in the classical theory. In those cases the reader is referred to the text-books on projective geometry (for example [2], [9], [10]).

Notation: We use logical symbols as abbreviations. And, as we do not intend to build a formalised theory in the classical sense (on the basis of intuitionistic logic), we shall not be too particular in using them. They must be understood in the intuitionistic sense [6, 7.1.1, 7.2.1]. $\rightarrow$ stands for implication, $\wedge$ for conjunction, $\vee$ for disjunction, $\neg$ for negation, $\bigwedge_{1 \leqslant i \leqslant n} p_i$ stands for $p_1 \wedge p_2 \wedge \ldots \wedge p_n$, $\bigvee_{1 \leqslant i \leqslant n} p_i$ for $p_1 \vee \ldots \vee p_n$.
$(\forall x)$ is the universal quantifier (for every x), $(\exists x)$ is the existential quantifier (there exists an x such that).

We shall freely use expressions like "P lies on l", "A, B, C form an affine triangle".

Notwithstanding the apparent drawback of the symbol "$\in$", we use it for the incidence relation.

The author wishes to express his gratitude to Professor HEYTING for his interest during the preparation of this study; much of his advice and his many suggestions determined its final shape.

1. *Isomorphism*

The classical notion of a one-to-one mapping can be used in intuitionistic mathematics. We shall say that a law f, designing to every element of a species A an element of a species B in such a way that equal elements correspond with equal elements, is a one-to-one mapping. If, however, A and B are species with an apartness relation ($\#$), then we may define a stronger notion of one-to-one mapping f as a law, designing to every element of A an element of B in such a way that elements lie apart from each other if and only if the corresponding elements lie apart from each other. In symbols it reads:

$$a = b \leftrightarrow f(a) = f(b) \qquad \text{(one-to-one)}$$

$$a \# b \leftrightarrow f(a) \# f(b) \qquad \text{(strongly one-to-one).}$$

It is easily seen that the second definition is strictly stronger than the first one.

For completeness we state the axioms for the apartness-relation (denoted by $\#$):

S_1 $\quad a \# b \rightarrow b \# a$

S_2 $\quad \neg\, a \# b \leftrightarrow a = b$

S_3 $\quad a \# b \rightarrow (Vc)\,(c \# a \vee c \# b).$

Corresponding to the two notions of one-to-one mapping there are also two notions of isomorphism. We shall use the strong notion of isomorphism, i.e. isomorphism with respect to the apartness relation, because it is the more important one of the two. Note that in some cases the two definitions are equivalent. For example: if f is an isomorphism in the weak sense from a field F onto field F', then f is an isomorphism in the strong sense.

Proof: As usual one ascertains $f(1) = 1$.

Let $a \# 0$, then a^{-1} exists and $1 = f(1) = f(a^{-1} \cdot a) = f(a^{-1}) \cdot f(a)$. We see that $f(a^{-1}) \cdot f(a) \# 0$. This implies $f(a) \# 0$ [6, p. 50]. Now if $a \# b$, then $(a - b) \# 0$. We just showed that this implies $f(a - b) \# 0$, or $f(a) \# f(b)$.

Remark that dealing with groups we can define an isomorphism in the strong sense as a mapping f with the property $ab \# c \leftrightarrow f(a) \cdot f(b) \# f(c)$. Compare definition 9.

2. *The projective plane* [7]

Let Π and Λ be disjoint species, $\in$ is a binary relation with domain Π and range Λ; $\#$ is a binary relation for which both domain and range are Π.

We call the elements of Π (Λ) points (lines). The relation $\#$ is the apartness relation and the relation $\in$ is called the incidence relation. Points and lines are denoted by capitals in italics and lower case italics respectively. The notation $l \cap m$ is used for the species of points which are incident with both l and m.

We first define two more relations:

Definition 1: $A\omega l$ if $(\forall B)(B \in l \to A \# B)$ (A lies *outside* l).

Definition 2: $l \# m$ if $(\exists A)(A \in l \wedge A\omega m)$.

Definition 3: A projective plane is an ordered quadruple $(\Pi, \Lambda, \in, \#)$ with the following properties (a) and (b).

(a) S_1, S_2, S_3.

(b) P_1 $\quad A \# B \to (\exists l)(A \in l \wedge B \in l)$

P_2 $\quad A \# B \wedge A \in l \cap m \wedge B \in l \cap m \to l = m$

P_3 $\quad l \# m \to (\exists A)(A \in l \cap m)$

P_4 $\quad A \# B \wedge A \in l \wedge B \in l \wedge C\omega l \wedge A \in m \wedge C \in m \to B\omega m$ (triangle axiom)

P_5 (i) $(\exists A)(\exists B)(A \# B)$

(ij) $(\forall l)(\exists A_1)(\exists A_2)(\exists A_3)(\bigwedge_{i \neq j} A_i \# A_j \wedge \bigwedge_i A_i \in l)$

(iij) $(\forall l)(\exists A)(A\omega l)$.

Definition 4: If $A \# B$, then the line l satisfying $A \in l \wedge B \in l$ is denoted by AB.

Definition 5: If $l \# m$, then the point A, satisfying $A \in l \cap m$ (which is unique) is denoted by $l \cap m$ (the use of $l \cap m$ will always be unambiguous).

It has been proved that the relation $\#$ between lines (definition 2) satisfies S_1, S_2, S_3 [4]. Therefore it is an apartness relation.

We denote the projective plane by $\mathfrak{P}(\Pi, \Lambda, \in, \#)$, or, if no ambiguity is possible, by $\mathfrak{P}$.

3. *The affine plane* [7]

Let Π, Λ, $\in$ and $\#$ be as in the preceding section. We use the definitions 2–5 and add to them:

Definition 6: $l\sigma m$ if $l \# m \wedge (\exists A)(A \in l \cap m)$ (l *intersects* m).

Definition 7: $l \parallel m$ if $(\forall A)(A \in l \to A\omega m)$ (l is *parallel* to m).

Definition 8: An affine plane is an ordered quadruple $(\Pi, \Lambda, \in, \#)$ with the properties (a) and (b):

(a) S_1, S_2, S_3.

(b) A_1 $\quad l \# m \wedge A\omega l \to (\exists p)(A \in p \wedge l \cap p = l \cap m)$

A_2 $\quad A \# B \wedge A \in l \cap m \wedge B \in l \cap m \to l = m$

A_3 $\quad l\sigma m \to (\forall p)\,((\exists A)\,(A \in p \cap l) \vee (\exists B)\,(B \in p \cap m))$

A_4 $\quad A \# B \wedge A \in l \wedge B \in l \wedge C\omega l \wedge A \in m \wedge C \in m \to B\omega m$ (triangle axiom)

A_5 $\quad P\omega l \wedge l \cap m = \phi \wedge P \in m \wedge Q \in l \to Q\omega m$

A_6 $\quad (\forall l)\,(\exists m)\,(l \parallel m)$

A_7 (i) There exists at least one line

(ii) $(\forall l)\,(\exists A_1)\,(\exists A_2)\,(\exists A_3)\,(\exists A_4)\,(\bigwedge_i A_i \in l \wedge \bigwedge_{i \neq j} A_i \# A_j)$

(iii) $A \# B \to (\exists l)\,(A \in l \wedge B\omega l)$

(iv) $A \in l \to (\exists m)\,(A \in m \wedge l \# m)$.

We remark that A_5 can be formulated as follows:

$$l \cap m = \phi \wedge m \# l \to l \parallel m.$$

If (in A_1) l and m intersect in a point B, then A_1 asserts the existence of AB. From A_1 and A_6 it follows that in the case $l \cap m = \phi$, there is a line through A, parallel to l. A_1 is stronger than these two assertions, since the existence of the line p is also ensured when it is unknown whether $l\sigma m$.

The axioms S_1, S_2, S_3 hold for the relation $\#$ between lines (def. 3), so this is an apartness relation [7, p. 163]. Remark that by definition the relation $\parallel$ is neither reflexive nor transitive.

The affine plane is denoted by $\mathfrak{A}\,(\Pi, \Lambda, \in, \#)$, or simply $\mathfrak{A}$.

Axiom A_3 can be strengthened in the following way:

Theorem 1: $l\sigma m \to (\forall p)\,(p\sigma l \vee p\sigma m)$.

Proof: $l\sigma m \to (\exists P)\,(P \in l \cap p) \vee (\exists Q)\,(Q \in p \cap m)$ (A_3)

Suppose $Q \in p \cap m$. There exists a point D on m so that $D \# Q$, $l \cap m\,(A_7)$. Consider the line d through D and parallel to p. By [7, lemma 7.1] we know $d\sigma l \vee d\sigma m$. Say $d\sigma l$ and $A = d \cap l$. $d \parallel p \to A\omega p$.

$$A\omega p \to l \# p.\ l\sigma d \to (\exists S)\,(S \in p \cap l) \vee (\exists T)\,(T \in p \cap d).$$

Since $p \parallel d$, we know $(\exists S)\,(S \in p \cap l)$, moreover $l \# p$, so $l\sigma p$. In the same way we deduct $p\sigma m$ from $d\sigma m$.

4. *Extension of isomorphisms*

Definition 9: Let $\mathfrak{A}_i\,(\Pi_i, \Lambda_i, \in_i, \#_i)$ $(i = 1, 2)$ be two affine planes. A pair of maps (φ_1, φ_2) is called an isomorphism if

(a) φ_1 maps Π_1 onto Π_2

(b) φ_2 maps Λ_1 onto Λ_2

(c) $P\omega_1\, l \leftrightarrow P^{\varphi_1}\omega_2 l^{\varphi_2}$.

For shortness we shall denote φ_1, $\in_i$, $\#_i$, ω_i respectively by φ, $\in$, $\#$, ω.

Lemma 1: $P \in l \leftrightarrow P^\varphi \in l^\varphi$.

Proof: Let $P \in l$. Suppose $P^\varphi \omega l^\varphi$. By definition $P\omega l$. This contradicts $P \in l$, so $\neg (P^\varphi \omega l^\varphi)$. Now by [7, lemma 2.2] $P^\varphi \in l^\varphi$ holds. Analogously one proves the other implication.

Lemma 2: $P \# Q \leftrightarrow P^\varphi \# Q^\varphi$.

Proof: Let $P \# Q$. By A_7 $(\exists l)(P \in l \wedge Q\omega l)$. By definition and lemma 1 $P^\varphi \in l^\varphi \wedge Q^\varphi \omega l^\varphi$. This implies with definition 1 $P^\varphi \# Q^\varphi$. Analogously for the inverse implication.

Lemma 3: $l \# m \leftrightarrow l^\varphi \# m^\varphi$.

Proof: See the proof of lemma 2.

Lemma 4: φ is an isomorphism if and only if $P \# Q \leftrightarrow P^\varphi \# Q^\varphi$ and $P \in l \leftrightarrow P^\varphi \in l^\varphi$.

Proof: Use definition 1.

Consider in a projective plane the species of points lying outside a given line l_∞ and the species of lines lying apart from l_∞. These species with the restrictions of the relations $\in$ and $\#$ form an affine plane (a so-called "affine subplane of the projective plane") except for the fulfilment of A_7 (ii). Since the following remains true if we replace A_7 (ii) by A_7^* (ii): $(\forall l)(\exists A)(\exists B)(A \# B \wedge A \in l \wedge B \in l)$, we shall weaken the axiom-system in this section. An incidental drawback of A_7 (ii) is the existence of certain projective planes without affine subplanes. We shall say that an affine plane $\mathfrak{A}$ can be extended to a projective plane $\mathfrak{P}$ if $\mathfrak{A}$ is isomorphic to an affine subplane of $\mathfrak{P}$. $\mathfrak{P}$ is called an extension of $\mathfrak{A}$. It is clear from the definition that if two affine planes $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic and $\mathfrak{A}_1$ possesses a projective extension, then $\mathfrak{A}_2$ possesses an extension.

Theorem 2: If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are affine subplanes of $\mathfrak{P}_1$ and $\mathfrak{P}_2$ respectively, and if φ is an isomorphism of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$, then φ can be extended to an isomorphism of $\mathfrak{P}_1$ onto $\mathfrak{P}_2$.

Proof: (a) Each point P of $\mathfrak{P}_1$ is incident with two lines of $\mathfrak{A}_1$, lying apart from each other. This is clear in the case where P is an affine point. In the general case we can find by P_5 (iij) l such that $P\omega l$. There are points A_1, A_2, A_3 on l (P_5 (ij)), such that $A_i \# A_j$ for $i \neq j$. $P\omega A_i A_j \wedge \wedge A_i \# A_j \rightarrow A_i \omega P A_j$, so $PA_i \# PA_j$ for $i \neq j$. Using S_3 we conclude that at least two of the lines PA_j $(i=1, 2, 3)$ lie apart from l_∞ and thus belong to $\mathfrak{A}_1$. Let a, b be two lines of $\mathfrak{A}_1$ such that $a \# b$ and $P = a \cap b$. Define the mapping $\varphi_1 : P^{\varphi_1} = a^\varphi \cap b^\varphi$. Remark that for affine points φ_1 and φ coincide.

(b) P^{φ_1} is independent of the choice of a and b. Let $P=a \cap b=a_1 \cap b$ and $a, a_1 \# b$. We shall show $a^\varphi \cap b^\varphi=a_1{}^\varphi \cap b^\varphi$. $a \# b \rightarrow a^\varphi \# b^\varphi$. $a^\varphi \# b^\varphi \rightarrow (\exists X')(X'=a^\varphi \cap b^\varphi)$. Likewise $(\exists Y')(Y'=a_1{}^\varphi \cap b^\varphi$. Suppose $X' \# Y'$. $X', Y' \in b^\varphi$. $b^\varphi \cap l'_\infty=B'$. $X' \# Y' \rightarrow B' \# X' \vee B' \# Y'$. If $B' \# X'$, then, considering $b^\varphi \# l'_\infty$, we see $X' \omega l'_\infty$. Likewise $B' \# Y'$ implies $Y' \omega l'_\infty$. So one of the points X' and Y' belongs to $\mathfrak{A}_2$. Consequently P belongs to $\mathfrak{A}_1$, but then $P^\varphi=a^\varphi \cap b^\varphi$ and $P^\varphi=a_1{}^\varphi \cap b^\varphi$, so $a^\varphi \cap b^\varphi= =a_1{}^\varphi \cap b^\varphi$. This contradicts our supposition, thus $\neg X' \# Y'$. And this implies by S_2 $X'=Y'$.

Subsequently suppose $P=a_1 \cap b_1=a_2 \cap b_2$ and $a_i \# b_i$ $(i=1, 2)$.

Definition: $(p, q) \sim (x, y) \equiv p^\varphi \cap q^\varphi=x^\varphi \cap y^\varphi \wedge p \# q \wedge x \# y$. One easily sees that $\sim$ is an equivalence relation.

$a_1 \# b_1 \rightarrow a_1 \# a_2 \vee b_1 \# a_2$

$a_1 \# a_2 \wedge a_1 \cap b_1=a_1 \cap a_2 \rightarrow (a_1, b_1) \sim (a_1, a_2)$ (see above)

$a_1 \# a_2 \wedge a_2 \cap b_2=a_1 \cap a_2 \rightarrow (a_2, b_2) \sim (a_1, a_2)$.

From these two lines it follows that $(a_1, b_1) \sim (a_2, b_2)$. Starting from $b_1 \# a_2$ we reach the same conclusion. We proved $a_1 \cap b_1=a_2 \cap b_2 \wedge a_1 \# b_1 \wedge \wedge a_2 \# b_2 \rightarrow a_1{}^\varphi \cap b_1{}^\varphi=a_2{}^\varphi \cap b_2{}^\varphi$. This justifies the definition of φ_1. We now define the extension of φ to the species of lines of $\mathfrak{P}_1$. Every line l in $\mathfrak{P}_1$ contains two points X and Y, lying apart from each other. Define $l^{\varphi_1}=X^{\varphi_1} Y^{\varphi_1}$. As was done above we can prove that l^{φ_1} is independent of the choice of X and Y.

(c) φ_1 is an isomorphism.

Let $P \omega l$. We first show that P or l is affine. There exists an affine line m, going through P. $m \# l$. Put $A=l \cap m$. $m \# l \rightarrow m \# l_\infty \vee l \# l_\infty$. $l \# l_\infty$ means: l is an affine line. $m \# l_\infty$, then $B=m \cap l_\infty$. $P \# A \rightarrow \rightarrow P \# B \vee B \# A$.

(i) $B \# A \wedge AB \# l_\infty \rightarrow A \omega l_\infty$. By definition 2: $l \# l_\infty$, so l is affine.

(ii) $P \# B \wedge PB \# l_\infty \rightarrow P \omega l_\infty$, so P is affine. We know now that one of the elements P or l is affine. We treat the cases separately. Write $P^{\varphi_1}=P'$ and $l^{\varphi_1}=l'$.

1. P' is an affine point.

There are two affine lines a' and b' through P', such that $a' \# b'$. $a' \# b' \rightarrow l' \# a' \vee l' \# b'$.

If $l' \# b'$, then $Q'=l' \cap b'$.

$S'=b' \cap l'_\infty$. Since P' is an affine point, we know $S' \# P'$. $P' \# S' \rightarrow \rightarrow Q' \# P' \vee Q' \# S'$.

(i) $Q' \# S' \wedge b' \# l' \rightarrow S' \omega l'$ and $S' \omega l' \rightarrow l' \# l_\infty$. So l' is an affine line and $P^{\varphi_1}=P^\varphi$, $l^{\varphi_1}=l^\varphi$. By the definition of φ $p' \omega l'$ holds.

(ii) $Q' \# P' \wedge b' \# l' \rightarrow P' \omega l'$.

2. l' is an affine line.

We know that l is also an affine line. Choose A, $B \in l$ so that $A \# B$ and A, $B \omega l_\infty$. If $a = PA$, $b = PB$, then $P = a \cap b$, where $a \# b$ and a, b are affine lines. Consequently $P^{\varphi_1} = a^{\varphi_1} \cap b^{\varphi_1}$. Now A^{φ_1}, B^{φ_1}, a^{φ_1}, b^{φ_1} are all affine elements and by lemma 2, 3:

$A^{\varphi_1} \# B^{\varphi_1}$, $a^{\varphi_1} \# b^{\varphi_1}$. $P\omega l \rightarrow l \# b$, so $l^{\varphi_1} \# b^{\varphi_1}$.
$A^{\varphi_1} \# B^{\varphi_1} \wedge l' \# b^{\varphi_1} \rightarrow A^{\varphi_1}\omega b^{\varphi_1}$. $A^{\varphi_1}\omega b^{\varphi_1} \rightarrow A^{\varphi_1} \# P'$.
$P' \# A^{\varphi_1} \wedge a^{\varphi_1} \# l' \rightarrow P'\omega l'$. This completes the proof.

Corollary: If an affine plane possesses a projective extension, then this extension is determined up to an isomorphism. We state another version of theorem 2: If the affine planes $\mathfrak{A}_1$, $\mathfrak{A}_2$ are isomorfic and if $\mathfrak{A}_1$ possesses a projective extension $\mathfrak{P}_1$, then $\mathfrak{A}_2$ possesses a projective extension $\mathfrak{P}_2$ and the isomorphism of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ can be extended to an isomorphism of $\mathfrak{P}_1$ onto $\mathfrak{P}_2$.

5. *The projective extension*

In [7] a construction has been given for the projective extension of an affine plane. We shall sketch the procedure.

Definition 10: If $l \# m$, then $\mathfrak{P}(l, m) = \{x \mid l \cap m = l \cap x \vee l \cap m = m \cap x\}$. $\mathfrak{P}$ is called a *projective point*. If $l\sigma m$, then $\mathfrak{P}(l, m)$ is the species of all lines, incident with $l \cap m$, in this case we say that $\mathfrak{P}$ is an affine point. If we want to distinguish affine elements from projective elements, we shall denote them by italics.

Definition 11: $l\omega\mathfrak{P}$ if $(\forall p)\,(p \in \mathfrak{P} \rightarrow l \# p)$ (l lies outside $\mathfrak{P}$).

Definition 12: $\mathfrak{A} \# \mathfrak{B}$ if $(\exists l)\,(l \in \mathfrak{A} \cap l\omega\mathfrak{B})$.

Theorem 3: The relation $\#$ between projective points is an apartness relation. [7, theorem 7].

Definition 13: If $\mathfrak{A} \# \mathfrak{B}$ then
$\lambda(\mathfrak{A}, \mathfrak{B}) = \{\mathfrak{C} \mid \mathfrak{A} \cap \mathfrak{B} = \mathfrak{A} \cap \mathfrak{C} \vee \mathfrak{A} \cap \mathfrak{B} = \mathfrak{B} \cap \mathfrak{C}\}$. is called a *projective line*.

Remark: Whenever $\mathfrak{P}(l, m)$ or $\lambda(\mathfrak{A}, \mathfrak{B})$ occurs, it is understood that $l \# m$, respectively $\mathfrak{A} \# \mathfrak{B}$.

If $l \in \mathfrak{A} \cap \mathfrak{B}$, then $\lambda(\mathfrak{A}, \mathfrak{B})$ is the species of all projective points, incident with l (this is the case if either of the projective points $\mathfrak{A}$, $\mathfrak{B}$ is affine).

Definition 14: $\mathfrak{P}$ is incident with λ if $\mathfrak{P} \in \lambda$.

Definition 15: $\mathfrak{A}\omega\lambda$ if $(\forall\mathfrak{P})\,(\mathfrak{P} \in \lambda \rightarrow \mathfrak{P} \# \mathfrak{A})$ ($\mathfrak{A}$ *lies outside* λ).

Definition 16: $\lambda \# \mu$ if $(\exists\mathfrak{P})\,(\mathfrak{P} \in \lambda \wedge \mathfrak{P}\omega\mu)$.

So far we have introduced new points and lines. Subsequently we must prove the axioms of the projective plane for the introduced species of projective points and lines and relations $\in$, $\#$.

P_1 is an immediate consequence of definition 13.

P_3 holds if both lines are affine. In its generality P_3 has as yet neither been proved nor refuted.

HEYTING [7, p. 169] postulated therefore:

A_8: $\mathfrak{A} \# \mathfrak{B} \wedge l\omega\mathfrak{A} \rightarrow (\exists\mathfrak{C})\ (l \in \mathfrak{C} \wedge \mathfrak{C} \in \lambda(\mathfrak{A}, \mathfrak{B}))$. Or in an equivalent formulation, avoiding quantification over projective points:

A_8': $p \# q \wedge r \# s \wedge l\omega\mathfrak{P}(p, q) \wedge r\omega\mathfrak{P}(p, q) \rightarrow$

$(\exists t)[t \# l \wedge (\mathfrak{P}(p, q) \cap \mathfrak{P}(r, s) = \mathfrak{P}(p, q) \cap \mathfrak{P}(t, l) \vee$

$\mathfrak{P}(p, q) \cap \mathfrak{P}(r, s) = \mathfrak{P}(r, s) \cap \mathfrak{P}(t, l))]$.

In a large number of cases A_8 can be proved. For example in the case of a desargian affine plane one can coordinatize the plane in a well-known way. By introducing homogeneous coordinates a projective extension is readily constructed an A_8 can be proved. The finite planes provide another class of examples.

6. *Proof of the triangle axiom*

The triangle axiom P_4 was proved in [7, p. 170] for the case that two of the considered projective points are affine and for one case that one of them is affine.

Two other cases were not proved and were introduced as the axioms A_9 and A_{10}. We shall prove them here, using $A_1 - A_7$ only.

Definition: A projective point $\mathfrak{P}(l, m)$ is an *improper point* if $l \parallel m$.

Lemma 5: If A is an affine point and $\mathfrak{P}(l, m)$ is an improper point, then $A \# \mathfrak{P}$.

Proof: From A_7 we can conclude that there are at least three lines a, b, c lying apart from one another, incident with A. Each two of these lines intersect in A. Using A_3 we see that at least two of them have a point in common with l. Say $P \in a \cap l$ and $Q \in b \cap l$. $a \# b \rightarrow l \# a \vee l \# b$, so we conclude $l\sigma a \vee l\sigma b$. Suppose $l\sigma a$. $l \cap m = \phi$ and $l\sigma a$ imply by A_3 $(\exists X)\ (X \in m \cap a)$. $l \cap a\omega m$ (def. 7), so $a \# m$ and even $a\sigma m$. In the same way we can prove $x \in \mathfrak{P}(l, m) \rightarrow a\sigma x$.

Lemma 6: If $l \in \mathfrak{A} \cap \mathfrak{B} \wedge \mathfrak{A} \# \mathfrak{B}$, then either $\mathfrak{A}$ or $\mathfrak{B}$ is affine.

Proof: If $\mathfrak{P}$ is the improper point of l, then $\mathfrak{A} \# \mathfrak{B} \rightarrow \mathfrak{A} \# \mathfrak{P} \vee \mathfrak{B} \# \mathfrak{P}$. Suppose $\mathfrak{A} \# \mathfrak{P}$, then $(\exists m)\ (m \in \mathfrak{A} \wedge m\omega\mathfrak{P})$. Choose $C \in m$. By A_1 there exists a line l', incident with C and $\mathfrak{P}$. $m\sigma l'$ holds, so by lemma 5 also $m\sigma l$. And this implies that $\mathfrak{A}$ is an affine point. Analogously for $\mathfrak{B} \# \mathfrak{P}$.

Lemma 7: $P \in \lambda(\mathfrak{A}, \mathfrak{B}) \rightarrow \lambda(\mathfrak{A}, \mathfrak{B})$ is an affine line.

Proof: $\mathfrak{A} \# \mathfrak{B} \rightarrow P \# \mathfrak{A} \vee P \# \mathfrak{B}$. Suppose $P \# \mathfrak{A}$, then $P \cap \mathfrak{A} = \mathfrak{A} \cap \mathfrak{B}$. $(\exists l)\ (l \in P \cap \mathfrak{A})$, so $(\exists l)\ (l \in \mathfrak{A} \cap \mathfrak{B})$. By [7, th. 9] $\lambda = l$.

Theorem 4 (A_9): $A \# \mathfrak{B} \wedge \mathfrak{C} \omega A\mathfrak{B} \rightarrow A \omega \lambda(\mathfrak{B}, \mathfrak{C})$.

Proof: If $\mathfrak{P}(l, m) \in \lambda(\mathfrak{B}, \mathfrak{C})$ and if $\mathfrak{Q}$ is the improper point of l, then $A \# \mathfrak{Q}$ (lemma 6). So $A \# \mathfrak{P} \vee \mathfrak{P} \# \mathfrak{Q}$. $\mathfrak{P} \# \mathfrak{Q} \rightarrow \mathfrak{P}$ is an affine point, so (lemma 7) λ is an affine line. Now one of the points $\mathfrak{B}$ and $\mathfrak{C}$ is affine, i.e. we have reduced the problem to [7, th. 13, th. 14]. So $A\omega\lambda$ holds, and in particular $A \# \mathfrak{P}$. We have proved $(\forall \mathfrak{P})\ (\mathfrak{P} \in \lambda(\mathfrak{B}, \mathfrak{C}) \rightarrow A \# \mathfrak{P})$, i.e. $A\omega\lambda(\mathfrak{B}, \mathfrak{C})$.

Theorem 5 (A_{10}): $\mathfrak{B} \# \mathfrak{C} \wedge A\omega\lambda(\mathfrak{B}, \mathfrak{C}) \rightarrow \mathfrak{B}\omega A\mathfrak{C}$.

Proof: The lines $A\mathfrak{B}$ and $A\mathfrak{C}$ are both affine (lemma 7), $\mathfrak{P}$ and $\mathfrak{Q}$ are the improper points of $A\mathfrak{B}$ and $A\mathfrak{C}$ respectively.

Let $\mathfrak{R} \in A\mathfrak{C}$. We must prove that $\mathfrak{R} \# \mathfrak{B}$. $\mathfrak{B} \# \mathfrak{C} \rightarrow \mathfrak{R} \# \mathfrak{B} \vee \mathfrak{R} \# \mathfrak{C}$. In the second case $\mathfrak{R} \# \mathfrak{C} \rightarrow \mathfrak{R} \# \mathfrak{Q} \vee \mathfrak{C} \# \mathfrak{Q}$ (i) $\mathfrak{R} \# \mathfrak{Q} \rightarrow \mathfrak{R}$ is an affine point R (lemma 6), so $R \# \mathfrak{P}$. $R \# \mathfrak{P} \rightarrow \mathfrak{B} \# \mathfrak{P} \vee \mathfrak{B} \# R$. $\mathfrak{B} \# \mathfrak{P} \rightarrow \mathfrak{B}$ is an affine point and then, by [7, th. 14] $\mathfrak{B}\omega A\mathfrak{C}$. In particular $\mathfrak{B} \# R$. (ii) $\mathfrak{C} \# \mathfrak{Q} \rightarrow \mathfrak{C}$ is an affine point C (lemma 6). Now by [7, th. 15] $\mathfrak{B}\omega A\mathfrak{C}$. In particular $\mathfrak{B} \# \mathfrak{R}$. We have proved $(\forall \mathfrak{R})\ (\mathfrak{R} \in A\mathfrak{C} \rightarrow \mathfrak{B} \# \mathfrak{R})$, i.e. $\mathfrak{B}\omega A\mathfrak{C}$.

In general it is not known, whether one of the points figuring in the triangle axiom is affine. However, using A_8, it can be proved [7, th. 17] that at least one of them is proper. So at this moment we need A_8 for the deduction of the (projective) triangle axiom.

7. *Proof of A_8 in a special case*

When formulating an incidence theorem in affine geometry one has to take special care of the existence of certain points and lines. These difficulties can be avoided by using the notions of projective point and projective line. This is done in the following incidence theorem, the so-called *axial theorem of Pappos*:

Let there be given the points A, B, C, A', B', C' and lines l, m, such that $A, B, C \in l$; $A', B', C' \in m$; $A \# B \# C \# A$; $A' \# B' \# C' \# A'$; $A, B, C\omega m$; $A', B', C'\omega l$ and $\mathfrak{P}(AC', A'C) \in BB'$, then $\mathfrak{P}(CB', BC') \in$ $\in \lambda(\mathfrak{P}(AC', CA'), \mathfrak{P}(AB', BA'))$.

Theorem 6: $A_1, \ldots, A_7$ and the axial theorem of Pappos imply A_8. We first prove the following lemmas:

Lemma 8: $\mathfrak{P} \# \mathfrak{Q} \wedge x\omega\mathfrak{P} \wedge x\omega\mathfrak{Q} \to (\exists B)\,(B \in x \wedge B\mathfrak{P} \# B\mathfrak{Q})$.

Proof: $\mathfrak{P} \# \mathfrak{Q} \to (\exists l)\,(l \in \mathfrak{P} \wedge l\omega\mathfrak{Q})$. $x\omega\mathfrak{P} \wedge l \in \mathfrak{P} \to (\exists A)\,(A \in x \wedge A\omega l)$. $A\mathfrak{P}\sigma x \to (\exists S)\,(S \in A\mathfrak{P} \cap l) \vee (\exists T)\,(T \in x \cap l)$.

In the first case $\mathfrak{P}$ is an affine point. Then $\lambda(\mathfrak{P}\mathfrak{Q})$ is an affine line m. Now there is an affine point B on x so that $B \# \mathfrak{R}(x, m)$. Using the triangle axiom one easily proves that $B\mathfrak{P} \# B\mathfrak{Q}$. In the second case $x\sigma l$. Choose $B = T$, then $B\mathfrak{P} \# B\mathfrak{Q}$.

Lemma 9: Let $\mathfrak{P}$, $\mathfrak{Q}$, x and B be as in lemma 8, $C' \in x$, $C' \# B$. Then $(\exists B')\,(B' \in \mathfrak{P}B \wedge B' \# B \wedge C'B'\omega\mathfrak{Q})$.

Proof: $(\exists B')\,(\exists B'')\,(B', B'' \in B\mathfrak{P} \wedge B \# B' \wedge B' \# B'' \wedge B'' \# B)$.

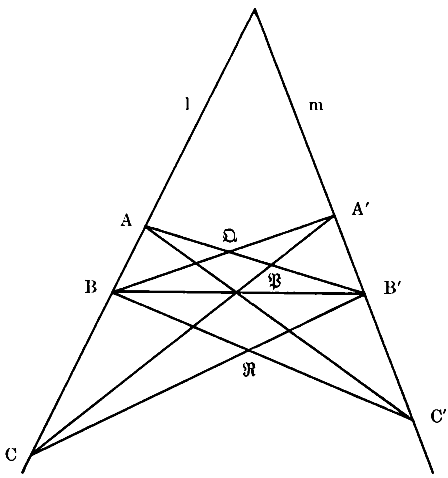
$C' \# B \wedge x \# B\mathfrak{P} \to C'\omega B\mathfrak{P}$. $C'\omega B\mathfrak{P} \wedge B' \# B'' \to C'B' \# C'B''$. Since $C' \# \mathfrak{Q}$, the line $C'\mathfrak{Q}$ exists.

$C'B \# C'B'' \to C'\mathfrak{Q} \# C'B' \vee C'\mathfrak{Q} \# C'B''$.

(i) $C'\mathfrak{Q} \# C'B \to (\exists X)\,(X \in C'B \wedge X\omega C'\mathfrak{Q})$. Applying the triangle axiom to the triangle $XC'\mathfrak{Q}$ (this is allowed since two of the points are affine) we find $\mathfrak{Q}\omega C'B$. This is equivalent to $C'B'\omega\mathfrak{Q}$.
Likewise we can prove

(ii) $C'\mathfrak{Q} \# C'B'' \wedge \mathfrak{Q} \# C' \to C'B''\omega\mathfrak{Q}$.

Thus a point with the desired properties can be found.



We now proceed to the proof of theorem 6. $\mathfrak{P}$ and $\mathfrak{Q}$ are projective points. $\mathfrak{P} \# \mathfrak{Q}$ and the affine line x satisfies $x\omega\mathfrak{P}$. We shall construct a projective point $\mathfrak{R}$, satisfying $x \in \mathfrak{R}$ and $\mathfrak{R} \in \lambda(\mathfrak{P}, \mathfrak{Q})$. We start with an extra assumption: $x\omega\mathfrak{Q}$.

We enumerate the steps of the construction.

a) Choose $B \in X$, so that $B\mathfrak{P} \# B\mathfrak{Q}$ (see lemma 8) (1)

b) Choose $C' \in x$, so that $C' \# B$ (2)

c) Choose $B' \in B\mathfrak{P}$ so that $B' \# B \wedge C'B'\omega\mathfrak{Q}$ (see lemma 9) (3)

d) Construct $C'\mathfrak{P}$

e) Construct $B'\mathfrak{Q}$

$x\omega\mathfrak{P} \rightarrow BB' \# x$

$BB' \# x \wedge C' \# B(2) \rightarrow C'\omega BB'$ (4)

$C'\omega BB' \rightarrow C' \# B'$ (5)

$C'\omega BB' \rightarrow C'\mathfrak{P} \# B'\mathfrak{P}$ (6)

$C'\mathfrak{P} \# B'\mathfrak{P} \wedge \mathfrak{P} \# B' \rightarrow B'\omega\mathfrak{P}C'$ (7)

$B'\omega\mathfrak{P}C' \rightarrow B'\mathfrak{Q} \# C'\mathfrak{P}$

Put $\mathfrak{A} = \mathfrak{P}(B'\mathfrak{Q}, C'\mathfrak{P})$

$\mathfrak{P} \# \mathfrak{Q} \rightarrow \mathfrak{A} \# \mathfrak{P} \vee \mathfrak{A} \# \mathfrak{Q}$

$\mathfrak{A} \# \mathfrak{P} \rightarrow \mathfrak{A}$ is an affine point or $\mathfrak{P}$ is an affine point.

In the last case $\lambda(\mathfrak{P}\mathfrak{Q})$ is an affine line, then $\mathfrak{R}$ exists trivially. So let us assume $\mathfrak{A}$ is an affine point. This implies $\mathfrak{A} = A \# \mathfrak{Q}$. Likewise we treat $\mathfrak{A} \# \mathfrak{Q}$. Thus we know that $\mathfrak{A}$ is an affine point A and $A \# \mathfrak{P}, \mathfrak{Q}$ or that one of the projective points $\mathfrak{P}, \mathfrak{Q}$ is affine. We assume $A \# \mathfrak{P}, \mathfrak{Q}$

$A \# \mathfrak{P} \wedge A \# \mathfrak{Q} \wedge A\mathfrak{P} \# A\mathfrak{Q} \rightarrow A\omega\lambda(\mathfrak{P}, \mathfrak{Q})$ (8)

$B'\mathfrak{P} \# C'\mathfrak{P}\ (6) \wedge B \# \mathfrak{P} \wedge A \# \mathfrak{P} \rightarrow A\omega\mathfrak{P}B,$ (9)

and $\mathfrak{P}\omega AB,$ (10)

and $A \# B$ (11)

f) Construct AB (11)

g) Construct $B'C'$ (5)

$C'\omega BB'\ (4) \wedge B \# B' \rightarrow B\omega m$ (12)

$C'B'\omega\mathfrak{Q}\ (3) \rightarrow AB' \# B'C'$

$AB' \# B'C' \wedge B' \# C' \rightarrow C'\omega AB'$ (13)

$B'\omega AC'\ (7) \rightarrow B' \# A$

$B' \# A \wedge C'\omega AB'\ (13) \rightarrow A\omega m$ (14)

$A\omega\mathfrak{P}B\ (9) \wedge B \# B' \rightarrow B'\omega l$ (15

$B\omega m\ (12) \rightarrow \mathfrak{Q}B \# m$

Put $\mathfrak{A}' = \mathfrak{P}(m, B\mathfrak{Q})$

$C'B'\omega\mathfrak{Q}$ (3) $\rightarrow AB'\sigma m$.

$AB'\sigma m \rightarrow (\exists X)\,(X \in AB' \cap B\mathfrak{Q}) \vee (\exists Y)\,(Y \in B\mathfrak{Q} \cap m)$.

In the first case we see that $\mathfrak{Q}$ is an affine point; then the theorem is correct. So we shall assume $(\exists Y)\,(Y \in B\mathfrak{Q} \cap m)$, i.e. $\mathfrak{A}'$ is an affine point A'.

$B\mathfrak{P} \# B\mathfrak{Q}$ (1) $\wedge B' \# B \rightarrow B'\omega BA'$ (16)

$B'\omega BA' \rightarrow A' \# B'$ (17)

$B'\omega l$ (15) $\rightarrow AB' \# l$.

$AB' \# l \rightarrow BA' \# l \vee AB' \# BA'$

(i) $BA' \# l \wedge A' \# B \rightarrow A'\omega l$ (18)

(ii) $BA' \# AB' \wedge A \# \mathfrak{Q} \wedge B \# \mathfrak{Q} \rightarrow \mathfrak{Q}\omega l$

$\mathfrak{Q}\omega l \rightarrow BA' \# l$.

$BA' \# l \wedge A' \# B \rightarrow A'\omega l$ (18)

In both cases we find the same result.

h) Construct $A'\mathfrak{P}$.

$A'\mathfrak{P} \# l$. Put $\mathfrak{C} = \mathfrak{P}(A'\mathfrak{P}, l)$

$\mathfrak{P}\omega l$ (10) $\rightarrow \mathfrak{P} \# \mathfrak{C}$

$\mathfrak{P} \# \mathfrak{C} \rightarrow \mathfrak{P}$ is an affine point or $\mathfrak{C}$ is an affine point.

Again we assume that $\mathfrak{C}$ is an affine point C.

$B'\omega\mathfrak{P}C'$ (7) $\wedge \mathfrak{P} \# C' \rightarrow \mathfrak{P}\omega m$ (19)

$\mathfrak{P}\omega m \rightarrow AC' \# m$

$A'\omega l$ (18) $\rightarrow A' \# C$

$A' \# C \wedge CA' \# m \rightarrow C\omega m$ (20)

$A' \# B'$ (17) $\wedge \mathfrak{P}\omega m$ (19) $\rightarrow B'\omega CA'$, $B'\omega CA' \rightarrow BB' \# CA'$

$BB' \# CA' \wedge \mathfrak{P} \# B \rightarrow B\omega CA'$

$B\omega CA' \rightarrow B \# C$. Likewise $A \# C$

$C'\omega BP$ (4) $\wedge B \# P \rightarrow B\omega C'A$

$A\omega m$ (14) $\rightarrow A \# C'$. $A \# C' \wedge B\omega C'A \rightarrow C'\omega l$

$x\omega\mathfrak{Q} \rightarrow x \# BA'$

$x \# BA' \wedge B \# C' \rightarrow C'\omega BA'$. $C'\omega BA' \rightarrow C' \# A'$.

i) Construct CB'

Then $\mathfrak{R} = \mathfrak{P}(BC', CB')$ is the point we looked for.

We still have to get rid of the extra assumption: $x\omega\mathfrak{Q}$.

We only assume now $l\omega\mathfrak{P}$. $(\exists A)\ (A \in l \wedge A\mathfrak{P} \# A\mathfrak{Q})$,

$(\exists m)\ (\exists n)\ (A \in m, n \wedge l \# m \# n \# l)$.

At least one of these lines, say m, lies apart from $A\mathfrak{P}$ and $A\mathfrak{Q}$.

$m \# A\mathfrak{P} \wedge A \# \mathfrak{P} \rightarrow m\omega\mathfrak{P}$, and likewise $m\omega\mathfrak{Q}$. We just showed the existence of $\mathfrak{R}$, so that $\mathfrak{R} \in m$ and $\mathfrak{R} \in \lambda(\mathfrak{P}, \mathfrak{Q})$.

$A\mathfrak{P} \# A\mathfrak{Q} \wedge A \# \mathfrak{Q} \rightarrow \mathfrak{Q}\omega A\mathfrak{P}$.

$\mathfrak{Q}\omega A\mathfrak{P} \wedge A \# \mathfrak{P} \rightarrow A\omega\lambda(\mathfrak{P}, \mathfrak{Q})$

$A\omega\lambda(\mathfrak{P}, \mathfrak{Q}) \rightarrow A \# \mathfrak{R}$ and $A \# \mathfrak{R} \wedge m \# l \rightarrow l\omega\mathfrak{R}$.

Now we can determine as we have done before $\mathfrak{R}'$ so that

$$\mathfrak{R}' \in l \wedge \mathfrak{R}' \in \lambda(\mathfrak{P}, \mathfrak{R}) = \lambda(\mathfrak{P}, \mathfrak{Q}).$$

This theorem shows that when trying to disprove A_8, we can leave out of consideration those planes in which the axial theorem of Pappos holds.

8. *The ternary field*

We shall give an intuitionistic treatment of the coordinatization procedure of HALL [2, 20.3], [10, 1.5]. Consider in the projective plane $\mathfrak{P}$ four points O, E, X, Y (lying apart from one another) so, that each point lies outside the lines determined by the other points. To each point P, outside XY, we adjoin the pair of points $(YP \cap OE,\ XP \cap OE)$. We call these the coordinates of P. We often shall identify P with this pair. Let l be a line, satisfying $Y\omega l$. If $(P_1, P_2) \in l$, then $P_2 = (P_1 Y \cap l)\ X \cap OE$.

We shall consider the affine subplane determined by XY and thus speak of "parallel". l is determined by the point $l \cap OY$ and the line l' through O and parallel to l. Say $N = (l \cap OY)\ X \cap OE$ and $M = (l' \cap EY)\ X \cap OE$, then $l = \{(MX \cap EY)\ O \cap XY\}\ (NX \cap OY)$. So

$$P_2 = (P_1 Y \cap \{(MX \cap EY)\, O \cap XY\}\ (NX \cap OY))\ X \cap OE.$$

On the species of affine points, incident with OE, is defined a ternary mapping by the formula above: $P_2 = \Phi(P_1, M, N)$. From this moment we denote the points of OE by lower case italics. O and E are denoted by 0 and 1.

Thus $y = \Phi(x, m, n)$ is the condition for a point (x, y) to be incident with a line $l = \{(mX \cap 1Y)\ 0 \cap XY\}\ (nX \cap 0Y)$. m and n are called the coordinates of the line l. We denote l by $[m, n]$. $y = \Phi(x, m, n)$ is the equation of l.

Theorem 7: $(x, y) \# (z, t) \rightarrow x \# z \vee y \# t$.

Proof: $(x, y) \# (z, t) \rightarrow (x, t) \# (x, y) \vee (x, t) \# (z, t)$. If $(x,t) \# (x,y)$, then the lines through (x, t) and X, (x, y) and X are parallel. This entails $t \# y$. Likewise $(x, t) \# (z, t)$ implies $x \# z$.

Theorem 8:

(a) $\Phi(0, m, n) = n$

(b) $\Phi(x, 0, n) = n$

(c) $\Phi(x, 1, 0) = x$

(d) $\Phi(1, m, 0) = m$

(e) If $\Phi(a, m, n) = b$, then n is uniquely determined by a, m, b.

(f) If $a_1 \# a_2$, $\Phi(a_1, m, n) = b_1$, $\Phi(a_2, m, n) = b_2$, then m and n are uniquely determined by a_1, a_2, b_1, b_2.

(g) If $m_1 \# m_2$, $\Phi(x, m_1, n_1) = \Phi(x, m_2, n_2)$, then x is uniquely determined by m_1, m_2, n_1, n_2.

(h) $\Phi(a, m, n) = \Phi(a, m', n') \wedge a \# a' \wedge m \# m' \rightarrow \Phi(a', m, n) \# \Phi(a', m', n')$

(i) $m \# 0 \wedge x_1 \# x_2 \rightarrow \Phi(x_1, m, n) \# \Phi(x_2, m, n)$.

(j) $x \# 0 \wedge m_1 \# m_2 \rightarrow \Phi(x, m_1, n) \# \Phi(x, m_2, n)$.

(k) $n_1 \# n_2 \rightarrow \Phi(x, m, n_1) \# \Phi(x, m, n_2)$.

(l) $\Phi(x_1, m_1, n_1) \# \Phi(x_2, m_2, n_2) \rightarrow x_1 \# x_2 \vee m_1 \# m_2 \vee n_1 \# n_2$.

Proof: (a) – (g) can be proved just as in classical mathematics, see [2], [9] or [10]. The proofs of (i) – (l) are quite straight forward. We shall only attend to (h). Put $p = \Phi(a, m, n)$, $q = \Phi(a', m, n)$, $r = \Phi(a', m', n')$ and consider the points $P = (a, p)$, $Q = (a', q)$, $R = (a', r)$. We shall prove that $Q \# R$. Let l_1 and l_2 be the lines through P with equation $y = \Phi(x, m, n)$, $y = \Phi(x, m', n')$ respectively. $m \# m' \rightarrow l_1 \sigma l_2$, for let M and M' be the intersections of l_1 and l_2 with XY. $m_1 \# m_2 \rightarrow M_1 \# M_2$. $M_1 \# M_2 \wedge \wedge P \omega XY \rightarrow M \omega PM'$ (triangle axiom), thus $l_1 \sigma l_2$.

$$a \# a' \rightarrow P \# Q. \; P \# Q \wedge l_1 \# l_2 \rightarrow Q \omega l_2. \; Q \omega l_2 \rightarrow Q \# R.$$

Finally $Q \# R \rightarrow q \# r$.

In the classical theory three classes of lines are introduced. Here we can do the same, but in general it need not be known to which class a line belongs.

Conclusion: The coordinatization is effective for:

(a)	affine points	$P - (p_1 \, p_2)$
(b)	improper points apart from Y:	$M - (m)$
(c)	Y:	$Y - (\alpha)$
(d)	lines, so that Y lies outside them:	$l - [m, n]$
(e)	affine lines through Y:	$l = [c]$
(f)	XY:	$[\beta]$.

α and β are mathematical objects which are not a member of T. We now proceed to the definition of a ternary field. Let T be a species with apartness relation ($\#$), 0 and 1 special elements of T and Φ a mapping of T^3 in T.

Definition 17: The ordered quintuple $(T, 0, 1, \#, \Phi)$ is a ternary field (with apartness relation) if (a)–(h), (k), (l) of theorem 8 hold [1]).

Remark 1: The properties (i) and (j) of a ternary field can be derived. (i). Consider the lines with equation $y=\Phi(x_1, m, n)$ and $y=\Phi(x, m, n)$.

$\Phi(x_1, 0, \Phi(x_1, m, n))=\Phi(x_1, m, n) \wedge x_1 \# x_2 \wedge m \# 0 \rightarrow \Phi(x_2, 0, \Phi(x_1, m, n)) \#$ $\# \Phi(x_2, m, n)$, thus by (b) $\Phi(x_1, m, n) \# \Phi(x_2, m, n)$.

(j) $\Phi(0, m_1, n)=\Phi(0, m_2, n) \wedge x \# 0 \wedge m_1 \# m_2 \rightarrow \Phi(x, m_1, n) \# \Phi(x, m_2, n)$.

Remark 2: The solutions in (e), (f), (g) are unique in a sharp sense (compare [6, 4.2.1]).

(e) $\Phi(a, m, n)=b \wedge n \# n' \rightarrow \Phi(a, m, n') \# b$ (by (k)).

(f) $(m \# m' \vee n \# n') \wedge a_1 \# a_2 \wedge \Phi(a_1, m, n)=b_1 \wedge \Phi(a_2, m, n)=b_2 \rightarrow$ $\rightarrow \Phi(a_1, m', n') \# b_1 \vee \Phi(a_2, m', n') \# b_2$.

Proof: Suppose $m \# m'$. Determine the unique elements n_1 and n_2, so that $\Phi(a_1, m', n_1)=b_1$ and $\Phi(a_2, m', n_2)=b_2$.

$\Phi(a_1, m', n_1)=\Phi(a_1, m, n) \wedge a_1 \# a_2 \wedge m \# m' \rightarrow \Phi(a_2, m', n_1) \# \Phi(a_2, m, n)$.

Thus $\Phi(a_2, m', n_1) \# \Phi(a_2, m', n_2)$.

By (l) $n_1 \# n_2$ holds. $n_1 \# n_2 \rightarrow n' \# n_1 \vee n' \# n_2$.

$n' \# n_1 \rightarrow \Phi(a_1, m', n') \# \Phi(a_1, m', n_1)$, or $\Phi(a_1, m', n') \# b_1$.

Likewise $n' \# n_2 \rightarrow \Phi(a_2, m', n') \# b_2$.

Next suppose $n \# n'$, then by (k) we see that $\Phi(a_1, m, n) \# \Phi(a_1, m, n')$. So $\Phi(a_1, m, n) \# \Phi(a_1, m', n')$ or $\Phi(a_1, m', n') \# \Phi(a_1, m, n')$. In the first case $\Phi(a_1, m', n') \# b_1$ holds. In the second case $m \# m'$ holds by (l). This last case we treated above.

(g) $x \# x' \wedge m_1 \# m_2 \wedge \Phi(x, m_1, n_1)=\Phi(x, m_2, n_2) \rightarrow \Phi(x', m_1, n_1) \#$ $\# \Phi(x', m_2, n_2)$.

The sharp uniqueness of the solution is expressed here by (h) itself. We can define in the usual way the two binary operations:

$$a+b=\Phi(a, 1, b)$$
$$a \cdot b=\Phi(a, b, 0).$$

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[1]) At first (a)–(g), (i)–(l) were used in definition 17. As for (h), a personal communication from Professor HEYTING drew special attention to it. He also noted the improvements indicated by remark 1 and 2.

We see that the affine points of the line OE (mentioned above) constitute a ternary field. This is the ternary field of the plane with respect to O, E, X, Y.

Definition 18: A mapping α of a ternary field T_1, onto a ternary field T_2 is an isomorphism if

$$y \# \Phi(x, m, n) \leftrightarrow y^\alpha \# \Phi(x^\alpha, m^\alpha, n^\alpha).$$

As an immediate consequence of this definition, we see

$$(\Phi(x, m, n))^\alpha = \Phi(x^\alpha, m^\alpha, n^\alpha).$$

If a ternary field is given we should expect a construction of a projective plane with an isomorfic ternary field. This construction fails, however, and the failure is due to the inhomogeneous way in which points and lines are introduced. We can obviously confine ourselves to the construction of an affine plane.

Definition 19: (a) a point is a pair of elements of T.

(b) a line is the species of points (x, y) which satisfy $y = \Phi(x, m, n)$ or the species of all points (x, y) with $x = c$ $(c \in T)$.

(c) $\in$ is the incidence relation.

(d) $(x, y) \# (z, t)$ if $x \# z \vee y \# t$.

Theorem 9: $(x, y)\omega[m, n] \leftrightarrow y \# \Phi(x, m, n)$.

Proof:

(a) $(x, y)\omega[m, n] \wedge (x, \Phi(x, m, n)) \in [m, n] \rightarrow (x, y) \# (x, \Phi(x, m, n))$.
By definition $y \# \Phi(x, m, n)$.

(b) $y \# \Phi(x, m, n) \wedge t = \Phi(z, m, n) \rightarrow t \# y \vee \Phi(x, m, n) \# \Phi(z, m, n)$.
In the last case by theorem 8 (l) $x \# z$.
So $(x, y) \# (z, t)$. This holds for all $(z, t) \in [m, n]$, so $(x, y)\omega[m, n]$.

Theorem 10: $(x, y)\omega[c] \leftrightarrow x \# c$.

Proof:

(a) $(x, y)\omega[c] \wedge (c, y) \in [c] \rightarrow (x, y) \# (c, y) \cdot (x, y) \# (c, y) \rightarrow x \# c$.

(b) $(z, t) \in [c] \rightarrow z = c$,
so $x \# c$ implies $(x, y) \# (z, t)$; this holds for all points of c, so $(x, y)\omega[c]$.

Theorem 11: $[m_1, n_1] \# [m_2, n_2] \leftrightarrow m_1 \# m_2 \vee n_1 \# n_2$.

Proof: Use theorem 9 and theorem 8 (l).

It is clear that we cannot affirm the existence of a line through (x, y) and (z, t) if it is not known whether $x \# z$. If we call, as usual, the improper point of the line with equation $x=0$ $(y=0)$ Y (resp. X), then we have so far constructed all (affine) lines l, so that $Y\omega l$ or $Y \in l$. In the case of general ternary fields no general construction for lines has been found, so we do not know whether every ternary field determines an affine plane. In the case of alternative fields or skew fields there are well known procedures to define in a homogeneous way points and lines of the projective plane. Since the axioms hold for these planes the problem is settled for these ternary fields.

Theorem 12: If T_1 and T_2 are ternary fields of the affine planes $\mathfrak{A}_1$ and $\mathfrak{A}_2$, then every isomorphism of T_1 onto T_2 can be extended to an isomorphism of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$.

Proof: We have coordinatized the affine planes in the indicated way: the points are determined by two coordinates, likewise the lines which intersect the line $0Y$ (lines of the first kind).

The lines, belonging to the projective point Y are given by one coordinate (lines of the second kind).

Let α be the isomorphism of T_1 onto T_2.

We define the mapping α_1: $(x, y)^{\alpha_1} = (x^\alpha, y^\alpha)$

$$[m, n]^{\alpha_1} = [m^\alpha, n^\alpha]$$

$$[c]^{\alpha_1} = [c^\alpha].$$

We remark that by theorem 9 and theorem 10 for a line of the first or the second kind the following holds:

$$P\omega l \leftrightarrow P^{\alpha_1}\omega l^{\alpha_1}.$$

We now define α_2 for the entire affine plane: $\alpha_2=\alpha_1$ for points. If l is a line of $\mathfrak{A}_1$ and there are P and Q so that $P \# Q$ and $P, Q \in l$, then $l^{\alpha_2} = P^{\alpha_1}Q^{\alpha_1}$.

(1) For lines of the first or second kind $\alpha_2=\alpha_1$ holds.

Let l be of the first kind: $l=[m, n]$.

$P \in l \leftrightarrow p_2 = \Phi(p_1, m, n)$.

$p_2 = \Phi(p_1, m, n) \leftrightarrow p_2{}^\alpha = \Phi(p_1{}^\alpha, m^\alpha, n^\alpha)$.

$p_2{}^\alpha = \Phi(p_1{}^\alpha, m^\alpha, n^\alpha) \leftrightarrow P^{\alpha_1} \in l^{\alpha_1}$.

So $P \in l \leftrightarrow P^{\alpha_1} \in l^{\alpha_1}$, likewise $Q \in l \leftrightarrow Q^{\alpha_1} \in l^{\alpha_1}$.

$P \# Q \rightarrow P^{\alpha_1} \# Q^{\alpha_1}$ (def. 19).

$P^{\alpha_1} \# Q^{\alpha_1} \wedge P^{\alpha_1} \in l^{\alpha_1} \wedge Q^{\alpha_1} \in l^{\alpha_1} \rightarrow P^{\alpha_1} Q^{\alpha_1} = l^{\alpha_1}$.

This shows $\alpha_1=\alpha_2$.

Let l be of the second kind: $l=[c]$.

$P \in l \leftrightarrow p_1 = c.$

$p_1 = c \leftrightarrow p_1^\alpha = c^\alpha.$

$p_1^\alpha = c^\alpha \leftrightarrow P^{\alpha_1} \in l^{\alpha_1}.$

So $P \in l \leftrightarrow P^{\alpha_1} \in l^{\alpha_1}$. Likewise $Q \in l \leftrightarrow Q^{\alpha_1} \in l^{\alpha_1}$. Again we see $l^{\alpha_2} = P^{\alpha_1} Q^{\alpha_1} = l^{\alpha_1}$, or $\alpha_1 = \alpha_2$.

(2) l^{α_2} is independent of the choice of P and Q. For lines of the first or second kind this is a direct consequence of the above.

Consider a line l of which it is unknown whether l is of the first or second kind.

$$P, Q_1, Q_2 \in l \wedge P \# Q_1 \wedge P \# Q_2.$$

Write $P^{\alpha_1} = P'$, $Q_i^{\alpha_1} = Q_i'$ and let y, y' be the lines $[0]$, $[0]^{\alpha_1}$. Suppose $P'Q_1' \# P'Q_2'$, then $P'Q_1' \sigma P'Q_2'$ and so, by theorem 1, $y' \sigma P'Q_1' \vee y' \sigma P'Q_2'$.

If $y' \sigma P'Q_1'$, then $y \sigma PQ_1$, i.e. $l = PQ_1$ is of the first kind. In that case $Q_2 \in PQ_1$, so $Q_2' \in P'Q_1'$. But $P'Q_1' \# P'Q_2' \wedge P' \# Q_2' \to Q_2' \omega P'Q_1'$. We have produced a contradiction.

Thus $P'Q_1' = P'Q_2'$. Let now P_1, P_2, Q_1, Q_2 be points on l, so that $P_i \# Q_i$. We need the following proposition:

$$l \# m \wedge A \in l \wedge B \in l \wedge A \# B \to A \omega m \vee B \omega m.$$

Proof: $l \# m \to [\exists P) (P \in m \wedge P \omega l)$.

$P \omega l \to P \# A \wedge P \# B$. So the lines AP and BP exist. $P \omega l \wedge A \# B \to$ $\to A \omega BP$. $A \omega BP \to AP \# BP$. $AP \# BP \to m \# AP \vee m \# BP$.

$m \# AP \wedge A \# P \to A \omega m$.

Likewise $B \omega m$ holds if $m \# BP$.

Denote again the image-points by accents.

Suppose $P_1' Q_1' \# P_2' Q_2'$.

By the above $P_2' \# Q_2'$ implies $P_2' \omega P'Q_1' \vee Q_2' \omega P_1'Q_1'$.

Say $Q_2' \omega P_1'Q_1'$. We see that the line $P_1'Q_2'$ exists and $P_1'Q_1' \# P_1'Q_2'$.

As we have already proved, this leads to a contradiction, so $P_1'Q_1' = P_2'Q_2'$.

(3) $P \omega l \leftrightarrow P^{\alpha_2} \omega l^{\alpha_2}$.

There are three points A_1, A_2, A_3 on l, lying apart from one another. $P \omega l$ implies that the lines PA_1, PA_2, PA_3 lie apart from one another. By applying theorem 1 we conclude that y intersects at least two of them, say $y \sigma PA_1, PA_2$. Then PA_1 and PA_2 are of the first kind.

$$l^{\alpha_2} = A_1^{\alpha_2} A_2^{\alpha_2}. \; A_2 \omega PA_1 \to A_2^{\alpha_2} \omega (PA_1)^{\alpha_2}. \; A_1 \# P \to A_1^{\alpha_2} \# P^{\alpha_2}.$$

Using the triangle axiom we find $P^{\alpha_2}\omega l^{\alpha_2}$. Analogously for the inverse implication. This finishes the proof.

Corollary: If a ternary field determines an affine plane $\mathfrak{A}$, then $\mathfrak{A}$ is determined up to an isomorphism.

Remark that we have not translated axiom A_1 into a property of the ternary field. It is to be expected that such a property must be joined to the properties of the ternary field.

Since, however, such a procedure does not help us to overcome the trouble with affine lines and since it would look rather clumsy we have refrained from adding it to the list of properties. Moreover, we do not need this property for the proof of theorem 12.

We should like to point out here that A_1 is a rather strong axiom from an affine point of view, and that its main task is found in the construction of the projective extension.

9. *Ordered projective planes*

The intuitionistic theory of ordered projective planes has been developed by HEYTING in [3] and in a number of (unpublished) lectures. The reduction of cyclical order to (linear) order is treated in [3, § 15].

Definition 20: An ordered projective plane is an ordered quintuple $(\Pi, \Lambda, \in, \#, |)$ (see def. 3), where $|$ is a relation (the separation relation) between collinear pairs of points, if the following holds:

(a) S_1, S_2, S_3

(b) $P_1, \ldots, P_5$

(c) C_1 $A_1, A_2 \mid A_3, A_4 \rightarrow \bigwedge_{i \neq j} A_i \# A_j$

C_2 $A, B \mid C, D \rightarrow B, A \mid C, D \wedge C, D \mid A, B$

C_3 $\bigwedge_{\substack{i \neq j \\ 1 \leqslant i, j \leqslant 4}} A_i \# A_j \wedge (\exists l)\left(\bigwedge_i A_i \in l\right) \rightarrow$

$\rightarrow A_1, A_2 \mid A_3, A_4 \vee A_1, A_3 \mid A_2, A_4 \vee A_1, A_4 \mid A_2, A_3$

C_4 $A, C \mid B, D \wedge A, D \mid C, E \rightarrow A, D \mid B, E$

C_5 $(\exists l)(\exists A_1) \ldots (\exists A_4)\left(\bigwedge_i A_i \in l \wedge \bigwedge_{i \neq j} A_i \# A_j\right)$

C_6 $A_1, A_2 \mid A_3, A_4 \wedge \bigwedge_i A_i \in l \wedge S \omega l \wedge S \omega m \wedge \bigwedge_i S A_i \cap m = B_i \rightarrow$

$B_1, B_2 \mid B_3, B_4.$

(order is invariant with respect to projection).

We mention some results, for the proofs the reader is referred to the original paper of HEYTING [3].

For future use we define the relation $\}$ between unordered point-pairs:

Definition 21: $A, B \} C, D$ if $A, B \# C, D$ and $\neg (A, B \mid C, D)$. The following theorem, well-known in classical projective geometry, holds here too:

Theorem 13: $A, B \} X, Y \wedge A, B \mid Y, Z \rightarrow A, B \mid X, Z.$

Proof: Remark that $A, B \mid Y, Z$ implies $Y \# Z$, so $X \# Y$ or $X \# Z$. In both cases we can give the usual proof.

Theorem 14: If A, B and C, D are harmonic pairs, then $A, B \mid C, D$ [3, p. 59].

Definition 22: Let P, Q and A lie apart from one another, then $\Sigma = \{B \mid A, B \mid P, Q\}$ is a *segment*.

It is well-known that P and Q ($P \# Q$) define exactly two segments Σ_1 and Σ_2 on PQ. Remark that $\Sigma_1 \cup \Sigma_2 \cup \{P, Q\} = l$ need not be true.

Theorem 15: Every segment contains (at least) countably many points, lying apart from each other.

We shall now describe the construction of an order relation on an affine line [3, p. 63]. Let P, Q and E be mutually apart. Σ_1 is the segment, determined by P, Q, to which E belongs, the other one is Σ_2.

We define: for $A, B \in \Sigma_1$ that $A < B$ if $P, B \mid A, Q$.
for $A, B \in \Sigma_2$ that $A < B$ if $P, A \mid B, Q$.
for $A \in \Sigma_2$, $B \in \Sigma_1$ that $A < B$.

We still have to consider those points for which it is unknown whether they lie apart from P.

Let X and Y be points of PQ, so that $X \# Y$ and $Q \# X, Y$. Then one of them lies apart from P, say $X \# P$. There exists a point R, lying apart from P, Q, X, Y. $R \in \Sigma_1 \vee R \in \Sigma_2$ [3, pag. 61]. R determines (like P) with Q two segments Σ_1' and Σ_2'.

If $R \in \Sigma_1$, then $\Sigma_1' = \{S \mid S, P \mid R, Q\}$,
if $R \in \Sigma_2$, then $\Sigma_2' = \{S \mid S, P \mid R, Q\}$.

It was proved that the order relations, defined by the couples P, Q and R, Q agree on $\Sigma_i \cap \Sigma_j'$.

We define the order relation between X and Y with respect to the couple R, Q. Thus we see that the order relation can be defined for all point-pairs, lying apart from each other and from Q.

Definition 23: We say that a binary relation $<$ on a species S is a pseudo-order relation if

1) $a<b \rightarrow \neg(b<a) \wedge a \neq b$
2) $a<b \wedge b<c \rightarrow a<c$
3) $\neg(a<b) \wedge \neg(b<a) \rightarrow a=b$
4) $a<b \rightarrow (Vc)(a<c \vee c<b)$.

Example: the natural order relation in the species of real numbers is a pseudo-order relation [6, p. 106].

It is easily seen that the relation, we just introduced is a pseudo-order relation.

10. *Order and the ternary field*

It is clear from the above that the species of elements of a ternary field T of an ordered projective plane is pseudo-ordered. Of the two possible pseudo-orderings we choose that in which $0<1$ holds. We shall study the influence of the pseudo-ordering on T.

Theorem 16: $n<n' \rightarrow \Phi(a,m,n)<\Phi(a,m,n')$ (compare [9, th. 7.3.1]).

Proof: We obtain $\varphi(n)=\Phi(a, m, n)$ from n by projecting thrice, according to the definition of Φ. Since the cyclical order is invariant under projection, either the order is preserved or reversed. It is sufficient to show for one pair of points c, d that $c<d$ implies $\varphi(c)<\varphi(d)$. We first show: $\Phi(a, m, p) = 0 \wedge \Phi(a, m, 0) = z \wedge z \# 0 \rightarrow p, z \mid O, W$ (where $W= XY \cap OE$).

Consider the line l with equation $y=\Phi(x, m, 0)$.

$R=l \cap XY$. Then $z=(0R \cap a\,Y)\,X \cap 0W$;

$O=\{R(X\,p \cap OY) \cap aY\}\,X \cap OW$, so we find

$p=\{(aY \cap OX)\,R \cap OY\}\,X \cap OW$.

Denote $OR \cap aY$ by S. Considering the quadrangle $OXYS$ we see that $U=OX \cap aY$, R, $V=OY \cap RU$, $T=XS \cap RU$ form a harmonic quadruple. So $U, R \mid V, T$. Projecting from X onto OW we see $O, W \mid p, z$.

Now $z \# 0 \rightarrow p \# 0$ and $p \# 0 \rightarrow p<0 \vee 0<p$

$0<p \rightarrow z<0$ or $\Phi(a, m, 0)<\Phi(a, m, p)$

$p<0 \rightarrow 0<z$ or $\Phi(a, m, p)<\Phi(a, m, 0)$.

This does not finish the proof, for we have still to consider the case in which it is unknown whether $a.m=\Phi(a, m, 0) \# 0$.

Let $x<0$, then $\varphi(0)<0 \vee \varphi(0)>x$.

(i) $\varphi(0)<0 \rightarrow \varphi(0) \# 0$, so *a.m.* $\# 0$. Then the theorem is correct.

(ii) $\varphi(0)>x \rightarrow \varphi(x)>x \vee \varphi(x)<\varphi(0)$
$\varphi(x)>x \rightarrow$ *a.m.* $\# 0$, as is clear by the definition of φ.

Again the theorem is correct. There remains $\varphi(x)<\varphi(0)$.
This finishes the proof.

Corollary: $n<n' \rightarrow x+n<x+n'$.

Theorem 17: $x<x' \rightarrow x+n<x'+n$.

The usual classical proof is applicable here.

Theorem 18: If the lines $[m_1, n_1]$, $[m_2, n_2]$ intersect in the point (s, t) and $m_1<m_2$, then

(a) $x>s \rightarrow \Phi(x, m_1, n_1)<\Phi(x, m_2, n_2)$

(b) $x<s \rightarrow \Phi(x, m_1, n_1)>\Phi(x, m_2, n_2)$.

Here too the classical proof can be used.
Next we give a definition of a pseudo-ordered ternary field.

Definition 24: A pseudo-ordered ternary field is an ordered quintuple $(T, 0, 1, \Phi, <)$, where $<$ is a pseudo-ordering on T, with the properties:

(1) (a)–(g), (l) of theorem 8.

(2) $n<n' \rightarrow \Phi(a, m, n)<\Phi(a, m, n')$.

(3) If $\Phi(s, m_1, n_1)=\Phi(s, m_2, n_2)$, then
$p>s \wedge m_1<m_2 \rightarrow \Phi(p, m_1, n_1)<\Phi(p, m_2, n_2)$ and
$p<s \wedge m_1<m_2 \rightarrow \Phi(p, m_1, n_1)>\Phi(p, m_2, n_2)$.

The absence of $\#$ in the quintuple need not disturb us, since we define $a \# b$ as $a<b \vee b<a$.
From (2) we readily conclude:

Corollary 1: $n<n' \rightarrow x+n<x+n'$.

Corollary 2: $n \# n' \rightarrow \Phi(a, m, n) \# \Phi(a, m, n')$.

An immediate consequence of (3) is:

Corollary 3: In a pseudo-ordered ternary field theorem 8, (*h*) holds.

The next theorems present no specifically intuitionistic difficulties, so we have omitted the proofs. For a treatment from the classical point of view the reader is referred to [9].

Theorem 19: $m_1 < m_2 \wedge x > 0 \rightarrow \Phi(x, m_1, n) < \Phi(x, m_2, n)$ and
$m_1 < m_2 \wedge x < 0 \rightarrow \Phi(x, m_1, n) > \Phi(x, m_2, n)$.

Corollary 1: $m_1 < m_2 \wedge x < 0 \rightarrow xm_1 > xm_2$ and
$m_1 < m_2 \wedge x > 0 \rightarrow xm_1 < xm_2$.

Corollary 2: $x \# 0 \wedge m_1 \# m_2 \rightarrow \Phi(x, m_1, n) \# \Phi(x, m_2, n)$.

Theorem 20: $x_1 < x_2 \wedge m > 0 \rightarrow \Phi(x_1, m, n) < \Phi(x_2, m, n)$ and
$x_1 < x_2 \wedge m < 0 \rightarrow \Phi(x_1, m, n) > \Phi(x_2, m, n)$.

Corollary 1: $x_1 < x_2 \wedge m > 0 \rightarrow x_1 m < x_2 m$ and
$x_1 < x_2 \wedge m < 0 \rightarrow x_1 m > x_2 m$.

Corollary 2: $x_1 \# x_2 \wedge m \# 0 \rightarrow \Phi(x_1, m, n) \# \Phi(x_2, m, n)$.

We easily derive:

Corollary 3: An ordered ternary field contains (at least) countably many elements, lying apart from each other.

We have showed by the way that the missing properties (h) and (k) of definition 17 are derivable here.

Theorem 21: If $\Phi(s, m_1, n_1) = \Phi(s, m_2, n_2)$ and $\Phi(p, m_1, n_1) < \Phi(p, m_2, n_2)$, then either $p > s \wedge m_1 < m_2$ or $p < s \wedge m_1 > m_2$.

11. *Extending order to the projective plane*

Starting from a pseudo-ordered ternary field T, belonging to a projective plane $\mathfrak{P}$, we shall introduce cyclical order in $\mathfrak{P}$. The divergence of our theory of the corresponding classical theory is for a considerable part due to the fact that it is seriously to be doubted whether a pseudo-ordering of an affine line can be extended to a cyclical ordering of its projective extension. Here is meant an affine line, considered as an independent structure, i.e. not imbedded in an affine plane. The problem of finding sufficient conditions for the problem, mentioned above is interesting in itself.

In this paper considerable use is made of the properties of the whole plane in order to define cyclical order on a projective line. Several times we shall need the existence of a point (line) lying apart from a number of points (lines). The existence is then ensured by theorem 26. We shall not fully demonstrate the existence each time.

Let $\mathfrak{P}$ be a projective plane. O, E, X, Y are chosen as usual. $\mathfrak{A}$ is the affine plane determined by XY and T is the ternary field of $\mathfrak{P}$ with respect to O, E, X, Y. The ternary field is pseudo-ordered.

Defining pseudo-order for affine lines we distinguish two cases.

Definition 25: Let A and B be points with coordinates (a_1, a_2) and (b_1, b_2), $A, B \in l$.

(i) if $Y\omega l$, then we define $A < B$ if $a_1 < b_1$

(ii) if $X\omega l$, then we define $A \lessdot B$ if $a_2 < b_2$.

Theorem 22: If $X, Y\omega l$, then

$$(\forall A)(\forall B)(A, B \in l \wedge A < B \to A \lessdot B) \vee$$
$$(\forall A)(\forall B)(A, B \in l \wedge A < B \to B \lessdot A).$$

Proof: If $X, Y\omega l$, then l is of the first kind, i.e. l has an equation $y = \Phi(x, m, n)$ and moreover $m > 0 \vee m < 0$.

Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $A < B \to a_1 < b_1$.
There are two cases:

$$m > 0 \wedge a_1 < b_1 \to \Phi(a_1, m, n) < \Phi(b_1, m, n),$$

or $a_2 < b_2$. Thus $A \lessdot B$.

$$m < 0 \wedge a_1 < b_1 \to \Phi(a_1, m, n) > \Phi(b_1, m, n),$$

or $a_2 > b_2$. Thus $A \gtrdot B$.

Thus we see that if both $<$ and $\lessdot$ are defined, they either coincide or are opposed to each other.

It would be natural now to introduce a betweenness relation in the affine plane. Since we are interested in the projective plane, we immediately pass to the next stage, i.e. the introduction of the separation relation.

Definition 26: $P_1, P_3 \mid P_2, P_4$ if $P_{\pi(1)} < P_{\pi(2)} < P_{\pi(3)} < P_{\pi(4)}$, where π is a permutation from the subgroup of $\mathfrak{S}_4$ generated by $\mathfrak{V}_4$ and $(1,3)$.

We also define the separation relation in the case that one of the four points is improper.

Definition 26a: If $Z \in XY$, then $P, R \mid Q, Z$ if $P < Q < R \vee R < Q < P$.

In these definitions we can also replace $<$ by $\lessdot$. In the next theorem it is asserted that these two pseudo-orderings provide the same cyclical ordering.

Theorem 23: If $<$ and $\lessdot$ are both defined on l, they define the same cyclical ordering.

Proof: Apply theorem 22 and definition 26 (26a).

Since at least one of the relations $<$ and $\lessdot$ is defined on a line l, we have defined the separation relation for all affine lines.

Theorem 24: The cyclical ordering $\mid$ is invariant under projection from any point Z of XY.

Proof: If $Z=Y$, then the theorem holds trivially.

(a) Next let Z be an improper point and $Z \# Y$. Then all lines through Z are of the first kind.

$Z \in l_1, l_2, l_1=[m, n_1], l_2=[m, n_2]$ and $n_1<n_2$.

Let l^* be a line of the first kind (i.e. $l^*=[m^*, n^*]$) and $m^* \# m$. l_1 and l_2 both intersect l^*:

$l^* \cap l_1=P=(p_1, p_2), l^* \cap l_2=Q=(q_1, q_2)$.

$$p_2=\Phi(p_1, m, n_1)=\Phi(p_1, m^*, n^*),$$

$$q_2=\Phi(q_1, m, n_2)=\Phi(q_1, m^*, n^*).$$

$n_1<n_2 \rightarrow \Phi(p_1, m, n_1) < \Phi(p_1, m, n_2)$.

$\Phi(q_1, m^*, n^*)=\Phi(q_1, m, n_2) \wedge \Phi(p_1, m^*, n^*)<\Phi(p_1, m, n_2) \rightarrow$

$\rightarrow (p_1>q_1 \wedge m^*<m) \vee (p_1<q_1 \wedge m^*>m)$. (theorem 21).

Thus, if $m^*>m$, then the points are in the same pseudo-order relation as that of n_1 and n_2. If $m^*<m$, then the relation between the points is opposite (we use $<$ for the points). So by projecting from Z from a line of the first kind onto a line of the first kind the pseudo-ordering is either invariant or reversed. In both cases the cyclical ordering is invariant.

(b) Let l^* be of the second kind: $l^*=[c]$.

$P=l^* \cap l_1 \rightarrow P=(c, p_2)$,

$Q=l^* \cap l_2 \rightarrow Q=(c, q_2)$.

$n_1<n_2 \rightarrow \Phi(c, m, n_1) < \Phi(c, m, n_2)$ or $p_2<q_2$.

This means $P \lessdot Q$. Here the pseudo-ordering is preserved, and therefore the cyclical ordering too.

(c) Let it be unknown whether l^* is of the first or second kind. $P=l^* \cap l_1$, $Q=l^* \cap l_2$. Since l_1 and l_2 are parallel, we know $P \# Q$. $P \# Q \rightarrow p_1 \# q_1 \vee p_2 \# q_2$. If $p_1 \# q_1$, then l^* is of the first kind and then the cyclical ordering of the points on l^* corresponds with the cyclical ordering of the lines through Z (being defined by the cyclical ordering of their second coordinates).

Consider $p_2 \# q_2$. Put $R=(p_1, \Phi(p_1, m, n_2))$.

$n_1<n_2 \rightarrow \Phi(p_1, m, n_1)<\Phi(p_1, m, n_2)$ or $p_2<r_2$

$p_2<r_2 \rightarrow q_2<r_2 \vee q_2>p_2$.

If $q_2<r_2$, then $q_2 \# r_2$. $q_2 \# r_2 \rightarrow \Phi(q_1, m, n_2) \# \Phi(p_1, m, n_2)$.

From this we conclude $q_1 \# p_1$. As before, this means that the cyclical ordering is invariant. Resuming we see that if $n_1<n_2$, the cyclical ordering is invariant or $q_2>p_2$, i.e. $P \lessdot Q$. In the last case the cyclical ordering is again invariant.

(d) Let it be unknown whether $Z \# Y$. We project from the line a onto the line b. $Z\omega a, b$, consequently $Z \# Y$ or a and b are of the first kind. We treated the case $Z \# Y$ above, and thus suppose that a and b are of the first kind.

$Z \in l_1, l_2$ and $l_1 \# l_2$. Both lines are affine. l_1 and l_2 intersect a and b. $P = l_1 \cap a$, $Q = l_2 \cap a$, $R = l_1 \cap b$, $S = l_2 \cap b$.

Let $P < Q$, or $p_1 < q_1$. As l_1 and l_2 are parallel, we see that $R \# S$. Since R and S are incident with a line of the first kind, this means $r_1 \# s_1$. $r_1 \# s_1 \rightarrow r_1 < s_1 \vee r_1 > s_1$.

Suppose $r_1 > s_1$. $p_1 < q_1 \rightarrow r_1 < q_1 \vee r_1 > p_1$.

$$r_1 < q_1 \wedge s_1 < r_1 \rightarrow s_1 < q_1.$$

So $p_1 < q_1 \rightarrow s_1 \# q_1 \vee r_1 \# p_1$.

In the first case we see that l_2 is of the first kind and in the second case l_1 is of the first kind. The problem is now reduced to $Z \# Y$, i.e. the cyclical ordering is invariant. Our conclusion is: $r_1 < s_1$ or the cyclical ordering is invariant. In the first case the cyclical ordering is evidently invariant too.

This finishes the proof of theorem 24.

Theorem 25: (i) If $A \in l$ and $<$ is the pseudo-order relation on l, then there exist points P and Q on l, so that $P < A < Q$.

(ii) If A, $B \in l$ and $<$ is the pseudo-order relation on l, and $A < B$, then there exists a point P on l, so that $A < P < B$.

Proof: Use theorem 20 corollary 3 and theorem 24.

Remark: We may replace in the theorem $<$ by $\dot{<}$.

Theorem 26: If A, B, and C lie apart from one another on an affine line l (one of them may be improper), then there exists a point P, so that $A, P \mid B, C$.

Proof: Use theorem 25.

One immediately verifies that the separation relation satisfies C_1 —— C_5. Though the verification can be a lengthy procedure it can every time be accomplished in a finite number of steps. We shall not go into this matter as it does not present any intuitionistic difficulties.

The following theorem is known as the axiom of Pasch. We have formulated it in terms of separation rather then in terms of betweenness.

Theorem 27: Let ABC be an affine triangle. $BC \cap XY = M$, $CA \cap XY = M'$, $AB \cap XY = M''$. l is a line so that A, B, $C\omega l$. $l \cap BC = P$, $l \cap CA = P'$, $l \cap AB = P''$. Then $B, C \mid P, M$ implies either $C, A \mid P', M'$ or $A, B \mid P'', M''$, but not both.

Proof: $B, C \mid P, M \rightarrow P \# M$. $P \# M \wedge BC \# XY \rightarrow P$ is an affine point. Then l is an affine line. $CA\sigma AB \rightarrow l\sigma CA \vee l\sigma AB$, i.e. one of the points P' or P'' is affine.

Suppose that P' is an affine point, then the points C, A, P', M' are apart from one another. Between the points C, A, P' some pseudo-order relation exists, consequently the four points can be divided into two separating pairs.

We consider two cases:

(i) $C, A \mid P, M'$.

Suppose $A, B \mid P'', M''$. From $P'' \# M''$ and $l \# P''M''$ we derive $M^* = l \cap XY \# M''$. Put $Q = M^*C \cap AB$. Projecting from M^*, we find by theorem 24: $Q, A \mid P'', M''$ and $Q, B \mid P'', M''$. Just as in classical geometry we derive a contradiction from these two propositions and $A, B \mid P'', M''$. So $A, B \wr P'', M''$.

(ii) $C, A \wr P', M'$. Again $l \cap XY = M^*$. Suppose for the moment $M^* \# M''$. Then $M^*C\sigma AB$, say $M^*C \cap AB = Q$. Projecting from M^* we find by theorem 24:

$$B, C \mid P, M \rightarrow B, Q \mid P'', M''; \quad C, A \wr P', M' \rightarrow Q, A \wr P'', M''.$$

Combining these two results we find $A, B \mid P'', M''$.

Now let it be unknown whether $M^* \# M''$. $AC\sigma AB \rightarrow M^*B\sigma AC \vee \vee M^*B\sigma AB$. If $M^*B\sigma AB$, then $M^* \# M''$ and this case we have already treated. So suppose $M^*B\sigma AC$. Projecting from M^* we find $M, P \mid B, C \rightarrow \rightarrow M', P' \mid B', C$ where $B' = M^*B \cap AC$.

By (ii) we know that $M', P' \wr A, C$ holds. $M', P' \wr A, C \wedge M', P' \mid \mid C, B' \rightarrow M', P' \mid A, B'$. This entails $A \# B'$. $A \# B' \wedge AC \# AB \rightarrow B'\omega AB$. $B'\omega AB \rightarrow M^*B \# M''B$. $M^*B \# M''B \wedge M'' \# B \rightarrow M''\omega M^*B$. $M''\omega M^*B \rightarrow M'' \# M^*$, here we are back again in the previous case. This finishes the proof.

We shall now define a new relation between pointpairs. First we define the relation for points lying apart from each other, afterwards we extend the definition to all pairs.

Definition 27: Let l be an affine line and let A and B be points, both lying outside l and XY, so that $A \# B$. If $AB \cap l = P$ and $AB \cap XY = M$ then $\delta_l^*(A, B)$ means $A, B \wr P, M$.

It is clear that $\delta_l^*(A, B) \rightarrow \delta_l^*(B, A)$. The relation δ_l^* is, however, neither reflexive nor transitive. To obtain a more suitable relation we proceed to

Definition 27a: Let l be an affine line and A and B points, both lying outside l and XY. Then $\delta_l(A, B)$ means

$$(\exists P)\,(\delta_l^*(A, P) \wedge \delta_l^*(B, P)).$$

The relation δ_l includes the relation δ_l^*. Let $\delta_l^*(A, B)$ hold, i.e. $A, B \wr P, M$. By a simple reasoning we derive from theorem 26 that there exists a point C, so that $A, B \mid C, M$.

$$A, B \mid C, M \wedge A, B \wr P, M \rightarrow A, B \mid P, C.$$

$$A, B \mid P, C \rightarrow A, C \wr P, B \text{ and } A, B \mid C, M \rightarrow A, C \wr B, M.$$

$$A, C \wr P, B \wedge A, C \wr B, M \rightarrow A, C \wr P, M, \text{ i.e. } \delta_l^*(A, C).$$

Likewise $\delta_l^*(B, C)$ holds, thus by definition: $\delta_l(A, B)$.

Theorem 28: δ_l is an equivalence relation on the species of all points outside l and XY.

Proof: (1) $\delta_l(A, A)$ is true.

Choose a line a through A. $a \cap l = P$, $a \cap XY = M$. By theorem 26 there exists a point B, so that $P, B \mid A, M$ consequently $P, M \wr A, B$. Thus by definition 27a $\delta_l(A, A)$ holds.

(2) $\delta_l(A, B) \rightarrow \delta_l(B, A)$ holds by definition.

(3) $\delta_l(A, B) \wedge \delta_l(B, C) \rightarrow \delta_l(A, C)$.

We prove (3) in a number of steps.

(3.1) $\delta_l^*(A, B)$ and $\delta_l^*(B, C)$ hold in the triangle ABC.
$BC \cap l = P$, $CA \cap l = P'$, $AB \cap l = P''$; $BC \cap XY = M$, $CA \cap XY = M'$, $AB \cap XY = M''$. By theorem 27 $A, B \wr P'', M'' \wedge B, C \wr P, M \rightarrow A, C \wr P', M'$, thus $\delta_l^*(A, C)$.

(3.2) $\delta_l^*(A, B)$ and $\delta_l^*(B, C)$ hold and A, B and C are collinear and lie apart from one other.

$$A, B \wr P, M \wedge B, C \wr P, M \rightarrow A, C \wr P, M, \text{ thus } \delta_l^*(A, C).$$

(3.3) Let the points A, B and C lie mutually apart. Let it be unknown whether $C \omega AB$ or not, then $\delta_l^*(A, C) \wedge \delta_l^*(B, C) \rightarrow \delta_l^*(A, B)$.

There exists a line m through C, so that $m \# AB$, AC, BC. Choose D and E on m, so that C, D and E lie apart from each other and $\delta_l^*(C, D)$, $\delta_l^*(C, E)$.

Then at least one of the points D and E lies outside AB (compare theorem 12), say $D \omega AB$. By these precautions D lies outside AB, BC, CA.

Apply (3.1) in triangle ACD: $\delta_l^*(A, C) \wedge \delta_l^*(C, D) \rightarrow \delta_l^*(A, D)$

and in triangle BCD: $\delta_l^*(B, C) \wedge \delta_l^*(C, D) \rightarrow \delta_l^*(B, D)$,

finally in triangle ABD: $\delta_l^*(A, D) \wedge \delta_l^*(B, D) \rightarrow \delta_l^*(A, B)$.

Remark: (3.3) tells us that $\delta_l(A, B) \wedge A \# B \rightarrow \delta_l^*(A, B)$.

(3.4) $\delta_l(A, B) \wedge \delta_l(B, C) \rightarrow \delta_l(A, C)$.

We can find P and Q, so that for P and Q holds: $\delta_l{}^*(A, P)$, $\delta_l{}^*(B, P)$, $\delta_l{}^*(B, Q)$, $\delta_l{}^*(CQ)$.

There exists a line m through P, so that $m \# CQ$. Choose $T \in m$, so that $T\omega CQ$, $T \# A$, $T \# B$ and $\delta_l{}^*(P, T)$. We now apply a number of times (3.3).

$$A \# T \wedge \delta_l{}^*(A, P) \wedge \delta_l{}^*(P, T) \rightarrow \delta_l{}^*(A, T).$$

$$B \# T \wedge \delta_l{}^*(B, P) \wedge \delta_l{}^*(P, T) \rightarrow \delta_l{}^*(B, T),$$

$$T \# Q \wedge \delta_l{}^*(B, T) \wedge \delta_l{}^*(B, Q) \rightarrow \delta_l{}^*(T, Q),$$

$$C \# T \wedge \delta_l{}^*(C, Q) \wedge \delta_l{}^*(T, Q) \rightarrow \delta_l{}^*(C, T).$$

$$\text{Finally } \delta_l{}^*(A, T) \wedge \delta_l{}^*(C, T) \rightarrow \delta_l(A, C).$$

This finishes the proof.

Theorem 29: If A and B lie outside l and XY then $\delta_l(A, B) \vee \vee \neg\, \delta_l(A, B)$.

Proof: Let $V \in XY$, so that $V \# l \cap XY$. Choose $P_1, P_2 \in AV$, with the properties $P_1 \# P_2$, $B \# P_1, P_2$ and $\delta_l{}^*(A, P_1)$, $\delta_l{}^*(A, P_2)$. Since $B \# P_1, P_2$, the lines P_1B, P_2B exist. $AV\sigma l \rightarrow P_1B\sigma AV \vee P_1B\sigma l$. If $P_1B\sigma l$, then $P_1B \cap XY \# P_1B \cap l$. If $P_1B\sigma AV$, then $B\omega AV$. Thus $BP_1\sigma BP_2$. This entails $BP_1\sigma l \vee BP_2\sigma l$. We have now ensured the existence of a point P with the properties $P \# A, B$, $\delta_l{}^*(P, A)$, $PB \cap l \# PB \cap XY$. Put $PA \cap l = S_1$, $PA \cap XY = M_1$, $PB \cap l = S_2$, $PB \cap XY = M_2$. By definition 27 either $\delta_l{}^*(P, B)$ or $\neg\, \delta_l{}^*(P, B)$ holds. If $\delta_l{}^*(P, B)$ holds, then $\delta_l(A, B)$ is true. If $\neg\, \delta_l{}^*(P, B)$ holds, then $P, B \mid S_2, M_2$. $\delta_l{}^*(A, P) \rightarrow \rightarrow A, P \wr S_1, M_1$. Projecting from the improper point T of l, we find $A', P \wr S_2, M_2$ (where A' is the projection of A). $A', P \wr S_2, M_2 \wedge P, B \mid \mid S_2, M_2 \rightarrow A', B \mid S_2, M_2$. This entails $A' \# B$. Choose $C \in AA'$, and let $A' \# C$. $C \# A' \cap BA' \# A'C \rightarrow C\omega BA'$. By the triangle axiom $B\omega AA'$ holds, thus $B \# A$. Then the line AB exists and projecting once more from T we find $B, A \mid S_3, M_3$ (where $S_3 = AB \cap l$, $M_3 = AB \cap XY$). This implies $\neg\, \delta_l{}^*(A, B)$. Above we showed that δ_l and $\delta_l{}^*$ coincide if $\delta_l{}^*$ is defined, so $\neg\, \delta_l(A, B)$ holds. This finishes the proof.

From the proof we also learn

Corollary: $\neg\, \delta_l(A, B) \rightarrow A \# B$.

Theorem 30: The relation δ_l determines exactly two equivalence-classes.

Proof: (a) theorem 26 entails that there are at least two equivalence-classes.

(b) It suffices by theorem 29 to prove

$$\neg\, \delta_l(A, B) \wedge \neg\, \delta_l(B, C) \rightarrow \delta_l(A, C).$$

By the corollary we know that $A \# B$, $B \# C$. Let m be a line through

A; $m \# AB$, BC. Choose D on m, such that $D\omega AB$, BC and $\delta_l{}^*(D, A)$. Applying the axiom of Pasch to the triangle ABD we find $\delta_l{}^*(B, D)$. Applying the axiom of Pasch once more to the triangle BCD we find $\delta_l{}^*(C, D)$. Finally $\delta_l{}^*(A, D)$ and $\delta_l{}^*(D, C)$ imply $\delta_l(A, C)$.

Theorem 31: Let P, Q, R, S be mutually apart, affine points on a line l, and T an affine point outside l, and $p = PT$, $q = QT$, $r = RT$, $s = ST$, then $P, R \mid Q, S \leftrightarrow (\delta_q(P, R) \wedge \neg\, \delta_s(P, R)) \vee (\delta_s(P, R) \wedge \neg\, \delta_q(P, R))$.

The classical proof is applicable [9, th. 7, 4.8]. For shortness we define a new relation. This relation is defined under rather restrictive conditions.

Definition 28: Let P, R, q, s be affine elements, so that $P \# R$, $q \# s$, $P, R\omega q, s$, then $\delta_{qs}(P, R)$ means $(\delta_q(P, R) \wedge \neg\, \delta_s(P, R)) \vee (\delta_s(P, R) \wedge$ $\wedge \neg\, \delta_q(P, R))$.

We reformulate the conclusion of theorem 31: $P, R \mid Q, S \leftrightarrow \delta_{qs}(P, R)$.

Theorem 32: Let P, Q, R, S, p, q, s, T be given as in theorem 30; Let $P' \in p$ and $P' \# T$, $P'\omega XY$, then $\delta_{qs}(P, R) \to \delta_{qs}(P', R)$.

Proof: $P' \# T \wedge p \# r \to P'\omega r$. $P'\omega r \to P' \# R$.

Suppose $\neg\, \delta_q(P, R) \wedge \delta_s(P, R)$. Put $M = p \cap XY$. As $P'\omega s$, we know that either $\delta_s(P, P')$ or $\neg\, \delta_s(P, P')$. We treat these cases separately:

(i) $\delta_s(P, P') \wedge \delta_s(P, R) \to \delta_s(P', R)$. (1)

It is clear that $\delta_s(P, P') \to \delta_q(P, P')$.

$$\delta_q(P, P') \wedge \neg\, \delta_q(P, R) \to \neg\, \delta_q(P', R) \qquad (2)$$

Combining (1) and (2) we find $\delta_{qs}(P', R)$.

(ii) $\neg\, \delta_s(P, P') \wedge \delta_s(P, R) \to \neg\, \delta_s(P', R)$ (3)

Again it is clear that $\neg\, \delta_s(P, P') \to \neg\, \delta_q(P, P')$

by theorem 29 $\delta_q(P, P') \wedge \delta_q(P, R) \to \delta_q(P', R)$ (4)

Combining (3) and (4) we find $\delta_{qs}(P', R)$.
Likewise we treat the case $\delta_q(P, R) \wedge \neg\, \delta_s(P, R)$.
Since in all cases we find $\delta_{qs}(P', R)$, the theorem is proved.

Theorem 33: If we project the affine points P, Q, R, S onto the affine points P', Q', R', S' from the affine centre T, then

$$P, R \mid Q, S \to P', R' \mid Q', S'.$$

Proof: Apply theorem 31 and theorem 32.

Remark: We can reformulate theorem 33, so that it holds even when some of the points are improper.

Now at last we are able to define a separation relation on all lines of P. Since the new relation will be an extension of the relation we introduced in definition 26 and 26a, we will denote it by the same symbol.

Definition 29: Let P, Q, R, S be four collinear points, lying apart from one another; l an affine line, so that $P, Q, R, S\omega l$; and T an affine point, so that $T\omega PQ$, l. Put $P' = PT \cap l, \ldots, S' = ST \cap l$. Then $P, R \mid Q, S$ if $P', R' \mid Q', S'$.

To justify this definition we remark that:

(i) the relation is independent of the choice of l (by theorem 33);

(ii) the relation is independent of the choice of T.

Proof: (a) In the first place we shall consider the improper points. An improper point Z is called "of the first kind" if $Z \# Y$. Suppose that $\bar{P}, \bar{Q}, \bar{R}, \bar{S}$ are improper points of the first kind. By projecting P', Q', R', S' (on the affine line l) onto XY from T we define the separation relation: $P', R' \mid Q', S' \to \bar{P}, \bar{R} \mid \bar{Q}, \bar{S}$. Let m_p, m_q, m_r, m_s be the first coordinates of the lines $T\bar{P}, T\bar{Q}, T\bar{R}, T\bar{S}$. A reasoning from classical geometry shows that $\delta_{pr}(Q, S) \leftrightarrow m_p, m_r \mid m_q, m_s$. Combining this with definition 29 and theorem 31 we find $\bar{P}, \bar{R} \mid \bar{Q}, \bar{S} \leftrightarrow m_p, m_r \mid m_q, m_s$. Now it is clear that the definition is independent of the choice of T.

If one of the points coincides with Y, then an analogous reasoning shows: $\bar{P}, \bar{R} \mid \bar{Q}, Y \leftrightarrow m_p < m_q < m_r \vee m_r < m_q < m_p$. Here the independence is also clear.

If it is unknown whether all points are of the first kind, we proceed as follows. At least three of the points are of the first kind, say $\bar{P}, \bar{Q}, \bar{R}$. Let there be given $\bar{P}, \bar{R} \mid \bar{Q}, \bar{S}$ (1). Projecting from an affine centre onto an affine line we find the affine points P'', Q'', R'', S''.

Suppose $P'', R'' \wr Q'', S''$ (2). We know that $\bar{P}, \bar{R} \mid \bar{Q}, Y$ (3) $\vee$ $\bar{P}, \bar{R} \wr \bar{Q}, Y$ (4). (1) and (4) imply $\bar{P}, \bar{R} \mid \bar{S}, Y$, thus $\bar{S} \# Y$. That case we considered before and led us to $P'', \bar{R}'' \mid \bar{Q}'', \bar{S}''$. This contradiction eliminates (4). (3) implies $P'', R'' \mid Q'', Y''$ (5) (where Y'' is the projection of Y). (5) and (2) entail $P'', R'' \mid S'', Y''$, thus $S'' \# Y''$. Then $\bar{S} \# Y$ also holds. We proved in that case that $P'', R'' \mid Q'', S''$, this contradicts (2), so $\neg (P'', R'' \wr Q'', S'')$ holds, i.e. $P'', R'' \mid Q'', S''$. This fully justifies definition 29 for improper points.

(b) If P, Q, R, S are affine then theorem 33 yields the independence. Let PQ be an affine line. Then at least three of the points are affine, say P, Q and R. $PQ \cap XY = U$. Projecting from a point T^* we find P^*, Q^*, R^*, S^* and U^* on l (remark that $TU\sigma l \vee TU\sigma TS$, in the last case $S \# U$, then by theorem 33 the independence is ensured). Considering $U' = TU \cap l$, we observe that either $P', R' \mid Q', U'$ or $P', R' \wr Q', U'$, in the last case $P', R' \mid Q', S'$ entails $P', R' \mid S', U'$, so $S' \# U'$. Then we know that S is an affine point, so the independence is proved. If

$P', R' \mid Q', U'$, then $P^*, R^* \mid Q^*, U^*$ (by theorem 33, remark). For P^*, Q^*, R^*, S^* the following holds: $P^*, R^* \mid Q^*, S^*$ or $P^*, R^* \wr Q^*, S^*$. Consider the last case: $P^*, R^* \wr Q^*, S^* \wedge P^*, R^* \mid Q^*, U^* \rightarrow P^*, R^* \mid S^*, U^*$. Then $S^* \# U^*$ holds, thus $S \# U$, i.e. S is affine. However, if S is affine, then we know that $P^*, R^* \wr Q^*, S^*$ is contradictory. We conclude that $P^*, R^* \mid Q^*, S^*$.

(c) Let it be unknown whether PQ is an affine line. P, Q, R, S, T and l are as in definition 29.

The improper points of $TP, \ldots, TS$ are $\bar{P}, \ldots, \bar{S}$. Project $P, Q, R, S, \bar{P}, \bar{Q}, \bar{R}, \bar{S}$ from a point T^* onto l, so that the images of P, Q, R, S are affine. A simple reasoning shows that either the images of $\bar{P}, \bar{Q}, \bar{R}, \bar{S}$ are affine or PQ is an affine line. The last case we considered above. The images are marked by an asterisk. We already know that $\bar{P}^*, \bar{R}^* \mid \bar{Q}^*, \bar{S}^*$ (1). The points P^*, Q^*, R^*, S^* lie apart from each other, so $P^*, R^* \mid Q^*, S^*$ (2) $\vee$ $P^*, R^* \wr Q^*, S^*$ (3). We shall derive a contradiction from the last formula. In the following reasoning we shall often use axiom S_3. If we encounter one of the formulae $P^* \# \bar{P}^*, \ldots, S^* \# \bar{S}^*$, we know that the line PQ is affine.

We then need only consider the other part of the disjunction. We frequently use this type of reasoning here, without fully motivating it.

We may assume that $P^* \# \bar{R}^*, \bar{Q}^*, \bar{S}^*$, so

$P^*, \bar{R}^* \mid \bar{Q}^*, \bar{S}^*$ (4) $\vee$ $P^*, \bar{R}^* \wr \bar{Q}^*, \bar{S}^*$ (5)

(1) and (5) entail $P^* \# \bar{P}^*$, so consider (4).

We may assume that $R^* \# \bar{Q}^*, \bar{S}^*$, so

$P^*, R^* \mid \bar{Q}^*, \bar{S}^*$ (6) $\vee$ $P^*, R^* \wr \bar{Q}^*, \bar{S}^*$ (7)

(4) and (7) entail $R^* \# \bar{R}^*$, so consider (6).

We may assume $Q^* \# S^*$, so $P^*, R^* \mid Q^*, \bar{S}^*$ (8)

or $P^*, R^* \wr Q^*, \bar{S}^*$ (9)

(6) and (9) entail $Q^* \# \bar{Q}^*$, so consider (8).

(8) and (3) entail $S^* \# \bar{S}^*$. This means that PQ is affine, and thus (3) is not true by (b).

We have now, by the way, advanced as far as the invariance of cyclical ordering with respect to projection from affine points. To complete the proof of axiom C_6 we still consider projections with improper centre and with a centre which is not known to be affine.

(α) The centre of projection is improper. It is clear that the range and domain of the projection are affine lines. We have already proved the invariance in the case that all points are affine, or in the case that one of them is improper. In general three of the points lie outside XY. Let it be unknown whether $S\omega XY$. Put $PQ \cap XY = U$, projecting from l onto m we find the points P', Q', R', S', U'. Let be given $P, R \mid Q, S$. We may suppose $P, R \mid Q, U$ (otherwise S would be affine, which we considered before). By theorem 24 $P', R' \mid Q', U'$. The points P', Q', R', S' are apart from one another and can be divided in separating pairs. Suppose $P', R' \wr Q', S'$, then $P', R' \mid Q', U'$ entails $U' \# S'$ thus S' is affine. As a consequence $P', R' \mid Q', S'$, which contradicts $P', R' \wr Q', S'$ so $P', R' \mid Q', S'$ holds.

(β) It is unknown whether the centre is affine or improper. We use here the same kind of reasoning as in (c) above, i.e. each time we encounter a disjunction in which one part affirms that some element is affine, we need consider the other part only. Let T be the centre of projection and l and m domain and range respectively. We remark that T is affine or l and m are affine. The last case remains to be considered. $P, Q, R, S \in l$; $P', Q', R', S' \in m$ and, as m is affine, at least three points are affine, say P', Q', R'. $PT \cap XY = T^*$. Projecting with centre T^* from l onto m we find $P^*, \ldots, S^*$ as images of $P, \ldots, S$. Let be given $P, R \mid Q, S$, we want to prove $P', R' \mid Q', S'$. By (a) $P^*, R^* \mid Q^*, S^*$ (1) holds. Suppose $P', R' \wr Q', S'$ (2).

We may assume $P' \# R^*, Q^*, S^*$, so $P', R^* \mid Q^*, S^*$ (3) $\vee$ $P', R^* \wr Q^*, S^*$ (4). (1) and (4) entail, $P' \# P^*$, so consider (3). We may assume $R' \# Q^*, S^*$, so $P', R' \mid Q^*, S^*$ (5) $\vee$ $P', R' \wr Q^*, S^*$ (6). (3) and (6) entail $R' \# R^*$, so consider (5). We may assume $Q' \# S^*$, so $P', R' \mid Q', S^*$ (7) $\vee$ $P', R' \wr Q', S^*$ (8). (5) and (8) entail $Q' \# Q^*$, so consider (7). Finally (2) and (7) entail $S' \# S^*$. A simple reasoning shows us that T' is affine if $S' \# S^*$. The results found above contradict (2), so $\neg\,(P', R' \wr Q', S')$, i.e. $P', R' \mid Q', S'$.

We have now fully established the invariance of cyclical order under projection.

Bearing in mind that the separation relation was introduced by means of the separation relation for affine points, it is obvious that the axioms $C_1 - C_5$ hold. The proof of C_6 we have just finished. By now we have reached our goal:

Theorem 34: The pseudo-ordering of a ternary field T of a projective plane $\mathfrak{P}$ can be extended to a cyclical ordering of $\mathfrak{P}$.

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Samenvatting

In dit proefschrift worden, van intuïtionistisch standpunt, een aantal uitbreidingsproblemen bestudeerd. Aanleiding hiertoe was het artikel [7] van Professor HEYTING, waarin de uitbreiding van een affien vlak tot een projectief vlak geconstrueerd werd. In de intuïtionistische affiene meetkunde kan men niet volstaan met de toevoeging van oneigenlijke punten en de oneigenlijke rechte, omdat ook die punten en lijnen toegevoegd moeten worden, waarvan het onbekend is of ze oneigenlijk zijn. Hier worden projectieve punten ingevoerd, naar het voorbeeld van [7], als waaiers van affiene lijnen (definition 10). Behalve de waaiers van snijdende lijnen en van evenwijdige lijnen komen hier bovendien waaiers voor, bestaande uit lijnen, waarvan het onbekend is of zij elkaar snijden.

Voor het bewijs van de axioma's van het projectieve vlak werden in [7] drie nieuwe axioma's ingevoerd. Twee daarvan zijn hier (theorem 4, theorem 5) afgeleid uit de axioma's van het affiene vlak. Het is niet bekend of het derde axioma (A_8) afgeleid kan worden uit de axioma's $A_1 - A_7$. Na toevoeging van de axiale stelling van PAPPOS aan het axiomastelsel blijkt A_8 afleidbaar te zijn (theorem 6). Hoewel het onzeker is of elk affien vlak een projectieve uitbreiding bezit, kunnen wij wel bewijzen dat een projectieve uitbreiding, indien deze bestaat, eenduidig bepaald is op isomorfie na (theorem 2, corollary).

Het tweede uitbreidingsprobleem behelst de constructie van een projectief vlak over een gegeven ternair lichaam. De axioma's van het ternaire lichaam met verwijderingsrelatie werden voor het eerst aangegeven door Professor HEYTING in zijn college "Intuïtionistische projectieve meetkunde". 1956–1957.

In de klassieke wiskunde voert men een aantal soorten punten en lijnen in (zie bij voorbeeld [2, p. 356]). Deze methode faalt echter bij een intuïtionistische behandeling. Men kan bij voorbeeld niet altijd uitmaken van welke soort de lijn is, die twee van elkaar verwijderde punten bepalen. Dit bezwaar doet zich hier reeds voor bij de constructie van een affien vlak over een ternair lichaam. De constructie van een projectief vlak is wel mogelijk langs algebraïsche weg als het ternaire lichaam een alternatief lichaam of een scheef lichaam is. Wanneer men aanneemt dat een ternair lichaam een affien vlak bepaalt, dan is dat vlak op, isomorfie na, eenduidig bepaald (theorem 12, corollary).

Het laatste deel van het proefschrift behandelt de ordening in een projectief vlak. In [3] werd de pseudo-ordening op de affiene rechte afgeleid uit de cyclische ordening van het vlak. Hier is, uitgaande van een pseudo-geordend ternair lichaam een cyclische ordening op het projectieve vlak gedefinieerd. In verband met het voorgaande was het daarbij noodzakelijk te veronderstellen dat het ternaire lichaam een projectief vlak bepaalt. Bij de behandeling van dit probleem werd gebruik gemaakt van het manuscript van [9], welwillend beschikbaar gesteld door de auteur.

STELLINGEN

I

Er bestaan voorbeelden van groepen, waarin de ontkenning van de ontkenning van de gelijkheid niet de gelijkheid inhoudt.

II

Men kan (groepen-)isomorfismen beschouwen ten opzichte van de verwijderingsrelatie, de gelijkheidsrelatie, de ongelijkheidsrelatie, Een isomorfisme van de eerste soort is strikt sterker dan een van de tweede soort en een isomorfisme van de tweede soort is strikt sterker dan een isomorfisme van de derde soort.

III

De karakteristiek van een intuïtionistisch lichaam is niet altijd te bepalen.

IV

Afgebakende puntsoorten behoeven geen diameter te bezitten. Begrensde afgebakende puntsoorten bezitten echter wel een diameter.

H. FREUDENTHAL. Zum intuitionistischen Raumbegriff. Comp. Math., 4, 1936, p. 99.

V

Men verkrijgt een generalisatie van het begrip ω-voudige negatieve convergentie door willekeurige deelsoorten der reële getallen toe te laten in plaats van een aftelbare rij getallen. De stellingen betreffende de meervoudige negatieve convergentie van som- en produktrijen kunnen bewezen worden. Bovendien geldt: is een rij $\{a_n\}$ negatief convergent naar twee deelsoorten, dan is $\{a_n\}$ negatief convergent naar de doorsnede.

J. G. DIJKMAN. Convergentie en divergentie in de intuïtionistische wiskunde. Dissertatie 1952, p. 13 e.v.

VI

Als F een complexe, lineaire ruimte van afbeeldingen van een verzameling X in een complexe, lineaire hausdorffruimte Y is, dan is er hoogstens een volledige, metrizeerbare, lineaire topologie waarvoor de afbeeldingen δ_x: $f \to f(x)$ van F in Y continu zijn.

VII

Laten P_1 en P_2 twee projectieve ruimten met homomorfisme zijn en laat P_2 een homomorf beeld van P_1 zijn. P_1 en P_2 kunnen dan zo gecoördinatizeerd worden m.b.v. lokale ringen R_1 en R_2, dat het homomorfisme van P_1 op P_2 geinduceerd wordt door een semi-lineaire afbeelding (f, σ), waarin σ een kanoniek homomorfisme is van R_1 op R_2.

W. KLINGENBERG, Projektive Geometrien mit Homomorphismus. Math. Ann. 132, 1956/57, p. 180–200.

VIII

De opmerking van McCALL, dat TARSKI en McKINSEY niet bewezen hebben, dat er juist zes formules in de intuïtionistische propositie calculus zijn, opgebouwd uit één variabele en "$\rightarrow$" en "$\neg$", is misleidend.

S. McCALL. A simple decision procedure for one-variable implicational negation formulae in intuitionistic logic. Notre Dame Journal of formal logic 1962, p. 120.
J. C. C. McKINSEY and A. TARSKI. Some theorems about the sentential calculi of Lewis and Heyting. Journal of symbolic Logic 1948, p. 1.

IX

Zowel tegen de argumenten waarmee P. G. J. VREDENDUIN betoogt dat "men bij de rechtvaardiging van de contrapositie door een bewijs uit het ongerijmde in een vicieuze cirkel geraakt" als tegen de conclusie zelf zijn bezwaren in te brengen.

P. G. J. VREDENDUIN. De contrapositie en het bewijs uit het ongerijmde. Euclides, 38, 1962, p. 20.

X

De procedure van KLINGENBERG, ter verkrijging van een isomorfisme van de partieel geordende verzameling der directe sommanden van een vrij moduul over een waarderingsring op het tralie der deelruimten van een vectorruimte, kan gegeneraliseerd worden.

W. KLINGENBERG. Projektive Geometrie und lineare Algebra über verallgemeinerten Bewertungsringen. Algebraical and topological foundations of geometry. Oxford 1962.

XI

Het zou de wiskunde-beoefening ten goede komen indien in ruimere mate in wiskunde-tijdschriften problemen gepubliceerd werden.

XII

Gezien de plaats, die de film zich in het culturele leven heeft verworven, is het wenselijk dat de filmkunst in het middelbaar onderwijs betrokken wordt.