Brouwer's Intuitionism: Philosophy, the Continuum, and boxing with your Feet

Why read Brouwer today?

The unintuitive nature of intuitionism

- **Theorem** (C-N) Every total real function is (ε, δ)-continuous
- $\neg\neg\neg p \rightarrow \neg p$ but not $\neg\neg p \rightarrow \neg p$
- And of course $p \vee \neg p$

Note that logical propositions are a product of intuitionism and not the base assumption

The intuition behind intuitionism – Brouwer's neointuitionism

However weak the position of intuitionism seemed to be after this period of mathematical development, it has recovered by abandoning Kant's apriority of space but adhering the more resolutely to the apriority of time. This neo-intuitionism

-Intuitionism and Formalism
L.E.J. Brouwer

Kant’s intuitionism

synthetic judgments *a priori* can be given. Geometry is based upon the pure intuition of space. Arithmetic attains its concepts of numbers by the successive addition of units in time, and pure mechanics especially can attain its concepts of motion only by employing the representation of time. Both representations, however, are merely intuitions; for if we omit from the empirical intuitions of bodies and their alterations (motion) everything empirical, i.e., belonging to sensation, space and time still remain, and are therefore pure intuitions that lie *a priori* at the basis of the empirical. Hence they

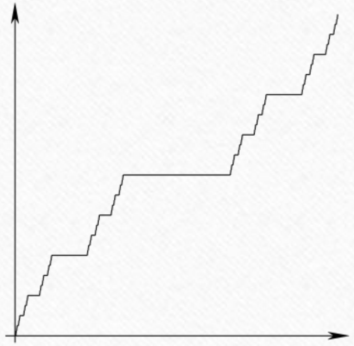
-Prolegomena to Any Future Metaphysics, §10
Immanuel Kant

But what is the basis of geometric intuition?

What is the line made of? Gallileo's Paradox
-Suppose we divide the staight 1meter line to infinite amount of equal size intervals $\{I_i\}^\infty$
-If the length of each interval is non-zero, then $\sum^\infty l(I_i) = \infty$
-If the length of each interval is zero, then $\sum^\infty l(I_i) = 0$

Can we find agreed upon the foundations of geometry?
-It was believed that the Euclidean point, line and plane are fundamental objects
-However, in the 19th century it was proven that they can be seen as a product of projective geometry's fundamental objects

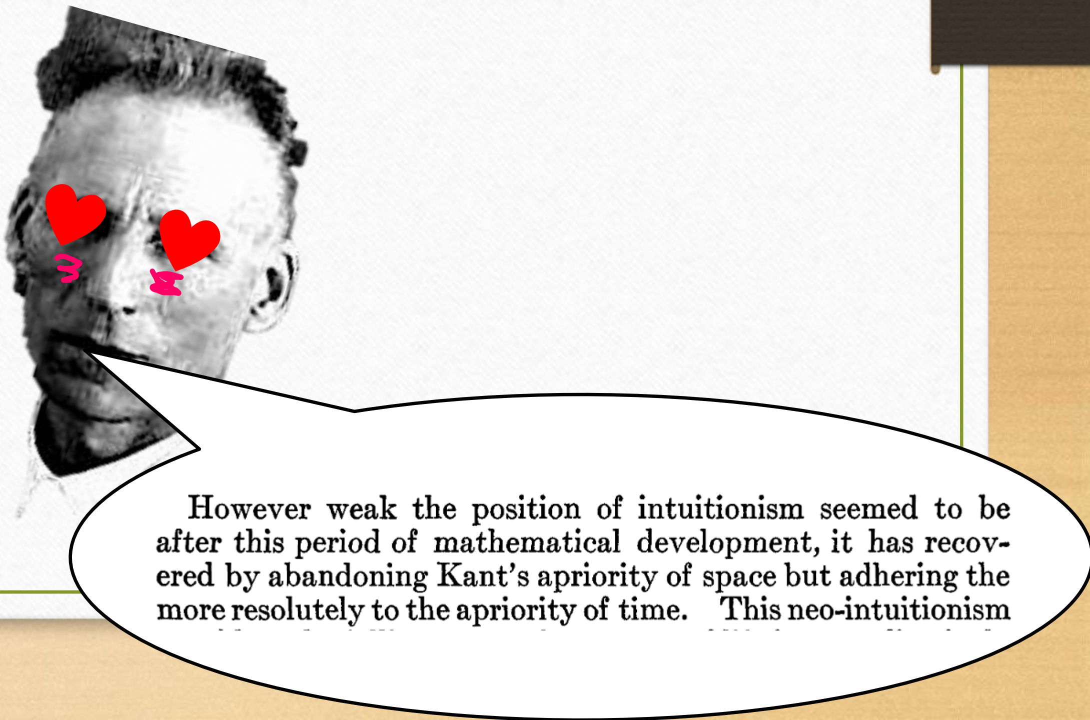
Geometric issuses are calculus issues:
-How to define the concept of continuity?
-And can Cantor stop breaking stuff: an increasing continuss function with zero derivative



Crisis in the foundation of geometry
Solution – Reduce geoemetry to arithmetics

In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continually, but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences. Even now such resort to geometric intuition in a first presentation of the differential calculus, I regard as exceedingly useful, from the didactic standpoint, and indeed indispensable, if one does not wish to lose too much time. But that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny. For myself this feeling of dissatisfaction was so overpowering that I made the fixed resolve to keep meditating on the question till I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis.

R. Dedekind, *Continuity and Irrational Numbers* (1872), pp. 1–2



But we know how the story went

- Arithmetization of the continuum required the Cantor-Dedekind axiom

> CANTOR-DEDEKIND THEOREM: *There exists an isomorphism between the set of real numbers and the set of points on a line in euclidean geometry.*
>
> -Fundamental Concepts of Geometry
> Bruce E. Meserve

- The reals were formulated on set theory
- The discovery of the Burali-Forti paradox raises doubt on the power of set theory as a foundation for mathematics
- The intuitionists and the formalists debated on how to interpret these results

Unlike Brouwer, 19th century trend was to completely reject intuition in mathematics and to focus on basic rules of inference

"Because intuition turned out to be deceptive in so many instances, and because propositions that had been accounted true by intuition were repeatedly proved false by logic, mathematicians became more and more skeptical of the validity of intuition. They learned that it is unsafe to accept any mathematical proposition, much less to base any mathematical discipline on intuitive convictions. Thus, a demand arose for the expulsion of intuition from mathematical reasoning, and for the complete formalization of mathematics. That is to say, every new mathematical concept was to be introduced through a purely logical definition; every mathematical proof was to be carried through strictly by logical means. The task of completely formalizing mathematics, of reducing it entirely to logic, was arduous and difficult; it meant nothing less than a reform in root and branch"

-The crisis in intuition
Hans Hahn, a notorious party pooper

Brouwer's main opponent – Hilbert's Formalism

"To make it a universal requirement that each individual formula then be interpretable by itself is by no means reasonable; on the contrary, a theory by its very nature is such that we do not need to fall back upon intuition or meaning in the midst of some argument. What the physicist demands precisely of a theory is that particular propositions be derived from laws of nature or hypotheses solely by inferences, hence on the basis of a pure formula game, without extraneous considerations being adduced."

-Hilbert 1927 (vH, 474-75)

Hilbert's Heaven

Ideal objects – axioms that maintain the inferences of our mathematics

↓ ↓ ↓

Objects of intuition – finitary objects – objects we want our mathematics to be consistent with

But where do the rules of inference come from? When do we consider a result ‘agreeable’?

- What is logic? Mathematics is the foundation of logic, logic is not the foundation of mathematics

Logic depends upon mathematics.
While thus mathematics is independent of logic, logic does depend upon mathematics: in the first place *intuitive logical reasoning* is that special kind of mathematical reasoning which remains if, considering mathematical structures, one restricts oneself to relations of whole and part; the mathematical structures themselves are in no respect especially elementary, so they do not justify any priority of logical reasoning over ordinary mathematical reasoning. It could be

- Logic is a social construct

“Therefore it is easily conceivable that, given the same organization of the human intellect and consequently the same mathematics, a different language would have been formed, into which the language of logical reasoning, well known to us, would not fit. Probably there are still peoples, living isolated from our culture, for which this is actually the case. And no more is it excluded that in a later stage of development the logical reasonings will lose their present position in the languages of the cultured peoples.”

The foundational crisis – Abuse of finitary reasoning

- We empirically observed regular patterns in our reasoning over finite amount of steps
- We concluded from them the syllogisms
- We attempted to apply these rules in an uninituitive manner on infinitary objects
- The irresponsible and unrestricted usage of mathematics led to the paradoxes

-On The Foundations of Mathematics

L.E.J. Brouwer

Brouwer’s Intuitionism

We should base mathematics on phenomenology and not axiomatization – observe the concepts we consider ‘intuitive’ and conclude from them

From the rejection of logic to “intuitionistic logic”

- For Brouwer, logic was a method of documenting mathematical constructions, but not rules for deducing new mathematical results
- Heyting was the one to formalize intuitionism into logic, Brouwer approved but saw it as a “pointless exercise”
- Proof interpetation of intuitionism – p is true if and only if p has a proof, ¬p is true if and only if p has a refutation

Is intuitionism to mathematicians what ornithology is to birds?

- Intuitionistic mathematics is often notorious for its creative, yet difficult to follow, and sometimes not even logically correct proves
- However, when considering the contribution of its founders to standard mathematics, it becomes a deep question of the correct methodology for the field