# Characterizations by nice forbidden sets

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**Cool Logic**

November 2nd, 2012

# Presentation guide

1. Introduction
2. Minimal
3. Antichain
4. Finite
5. A couple of questions

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# Graphs

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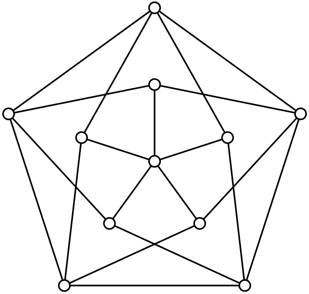
- Elegant and intuitive models of relations
- Many (!) applications in Mathematics, Logic, Computer Science, Physical/Biological/Social systems, ...
  - In 2012, both the Nobel Prize in Economics (A. Roth and L. Shapley) and the Abel Prize (E. Szemerédi) were given for work in Graph Theory!
- Nice visual representations:



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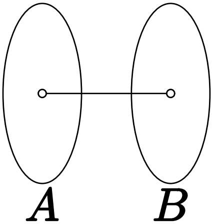

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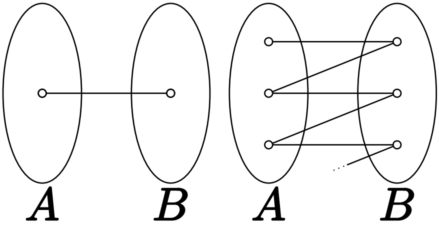

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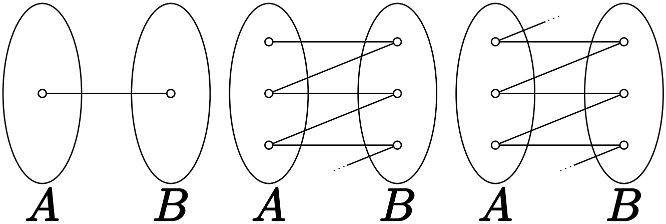

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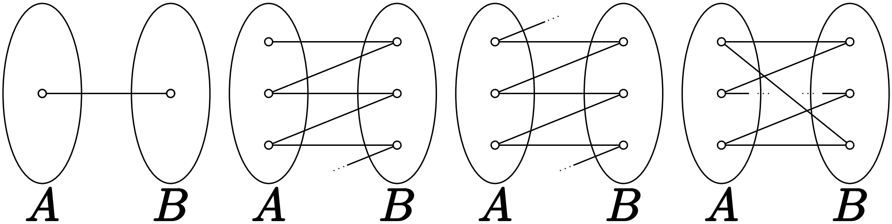

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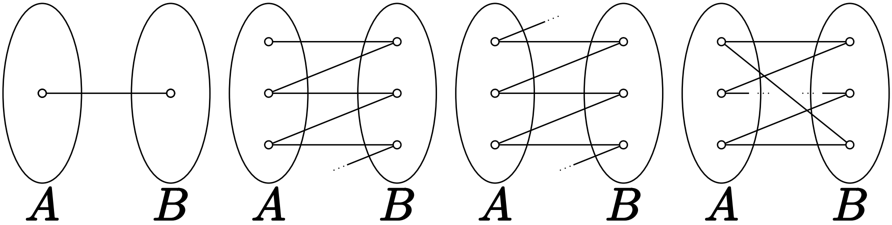
In any case you can choose edges so that each vertex is contained in exactly one chosen edge (this is called a perfect matching). ∎



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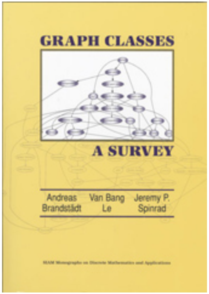
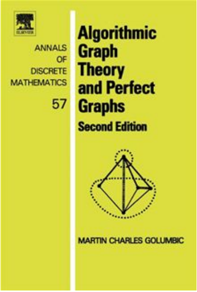
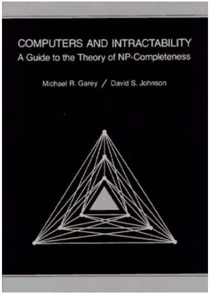
- Sometimes these properties can help solve problems we are interested in.
  - Underlying our last proof: "If the maximum degree of $G$ is at most 2 and no component of $G$ has odd cardinality, then $G$ has a perfect matching".
- Hence, a <u>lot</u> of focus is placed on studying specific classes of graphs.



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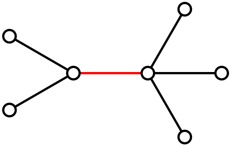

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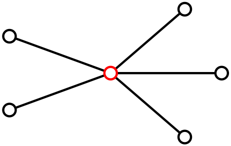

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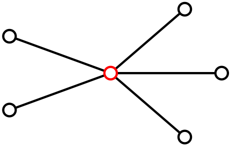

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(These are partial orders when the graphs are finite, but not in general)

# The main character enters the scene

One of the more common ways of characterizing (or defining) a class of graphs is using a forbidden set:

- A set $\mathscr{F}$ is a forbidden set (FS) for a class $\mathscr{C}$ when, for any graph $G$, we have

$$G \in \mathscr{C} \iff G \text{ does not contain any } H \in \mathscr{F},$$

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  where "containment" is according to the relation we are interested in.

- Intuitively, when $\mathscr{F}$ is nice in some way, this can say a lot about $\mathscr{C}$.
  - For instance, in the example we saw, the fact that $G$ contained no odd cycles was quite crucial.

# But when can you do this?

However, not every class of graphs has a FS.

- In fact, for any set $X$ in any preordered set $\langle \mathscr{P}, R \rangle$, we have that

$$X \text{ has a FS} \iff X \text{ is downwards-closed w.r.t. } R,$$

but in the general case we can only prove that $\overline{X}$ is a FS for $X$.

With graphs, in a sense we can always do better, with nicer FS.

# Making sense

One notion of "niceness" could be a type of minimality:

- $\mathscr{F}$ is a minimal FS when anything strictly below elements of $\mathscr{F}$ is not forbidden.
  - These are the FSs used in finite Graph Theory; indeed $\mathscr{F} = \{G \notin \mathscr{C} : G - v \in \mathscr{C} \text{ for all } v\}$, but determining $\mathscr{F}$ is ad hoc.
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But niceness can appear in other shapes, too (and sometimes it has to).

We will focus on 3 such shapes — forbidden sets which are...

- Minimal;
- Antichains;
- Finite.

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But not so for infinite graphs; the class of graphs with finitely many edges is closed under $\leq$, but cannot have a minimal FS.

# Characterization

It is easy to see that well-foundedness of the preordered set is not necessary.

In fact, not even well-foundedness of $\langle \overline{X}, R \cap \overline{X}^2 \rangle$ is necessary in order for $X$ to have a minimal FS.

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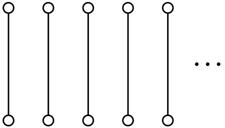

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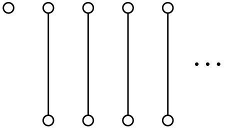

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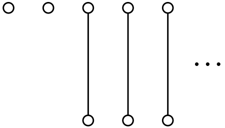

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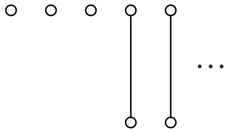

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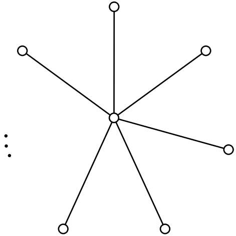

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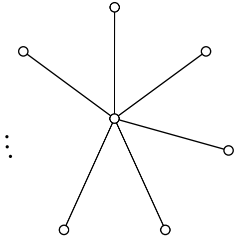

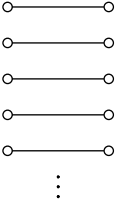

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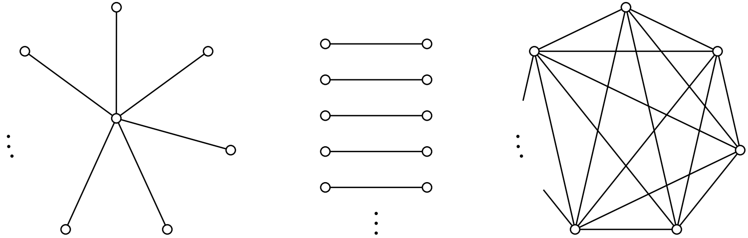

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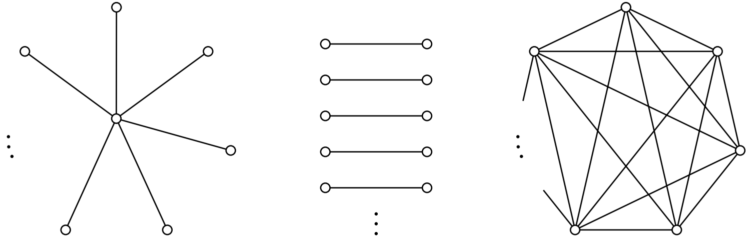

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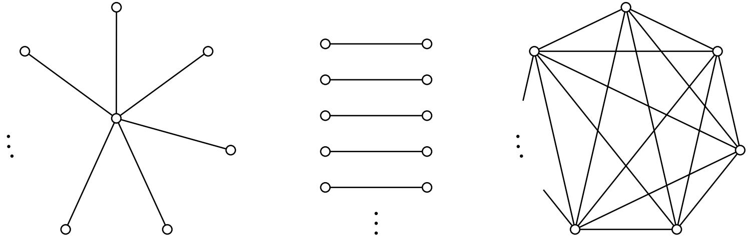

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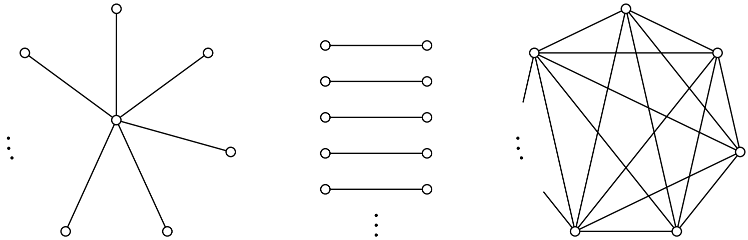

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As we will see, for minors a lot more holds.

# Quotienting away your troubles

Given a preordered set $\langle \mathscr{P}, R \rangle$, define an equivalence relation $\sim$ on $\mathscr{P}$ by

$$x \sim y \quad :\Longleftrightarrow \quad xRy \text{ and } yRx.$$

Notation:

- $[x]$ : the equivalence class of $x \in \mathscr{P}$;
- $[X] := \{[x] : x \in X\}$, for $X \subseteq \mathscr{P}$;
- $[R]$ : the partial order given by $[x][R][y]$ iff $xRy$.

# Characterization

**Theorem.**

$$X \text{ has a forbidden antichain in } \langle \mathscr{P}, R \rangle$$
$$\Longleftrightarrow$$
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Proof.

($\Longrightarrow$) If $F$ is a forbidden antichain for $X$, then $[F]$ is a forbidden antichain for $[X]$. But $\langle [\mathscr{P}], [R] \rangle$ is a poset, so $[F]$ is a minimal FS.

($\Longleftarrow$) If $[F]$ is a minimal FS for $[X]$, then let $F'$ be composed of exactly one element from each $[x] \in [F]$.
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In the proof of ($\Longleftarrow$) we made a clear use of **AC**.

Indeed, this use was essential...

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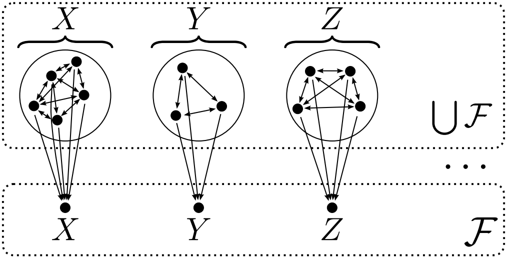
Define $\mathscr{P} := \left(\bigcup \mathcal{F}\right) \cup \mathcal{F}$, and let $R$ be the preorder on $\mathscr{P}$ given by

$$xRy \quad :\Longleftrightarrow \quad x = y, \text{ or } y \in x \in \mathcal{F}, \text{ or } \exists X \in \mathcal{F} \text{ such that } x, y \in X.$$



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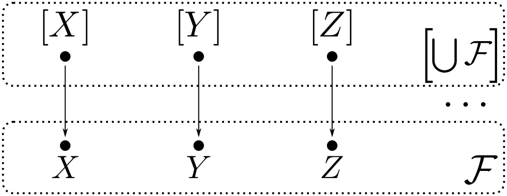
$$xRy \quad :\Longleftrightarrow \quad x = y, \text{ or } y \in x \in \mathcal{F}, \text{ or } \exists X \in \mathcal{F} \text{ such that } x, y \in X.$$

Now, $[\bigcup \mathcal{F}] \approx \{[X] : X \in \mathcal{F}\}$ is a minimal FS for $[\mathcal{F}] \approx \mathcal{F}$ in $\langle [\mathscr{P}], [R] \rangle$.



# Alwayz into somethin'...

**Theorem.**

($\Longleftarrow$) of the last Theorem is equivalent to **AC**.

Proof.

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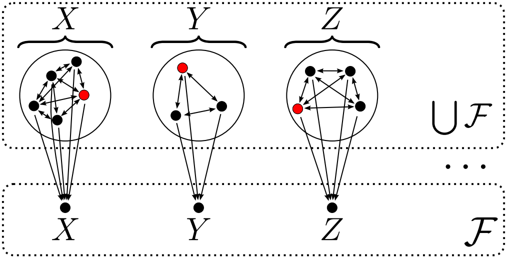
Thus $\mathcal{F}$ has a forbidden antichain in $\langle \mathscr{P}, R \rangle$, i.e., $\mathcal{F}$ has a choice set. ■



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2. Minimal
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# Graph Minor Theorem

The relation of graph minor came into the spotlight with (K. Wagner's version of) K. Kuratowski's theorem:

**Theorem (K. Kuratowski 1930, K. Wagner 1937).**

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One of the most celebrated recent developments in Graph Theory is:

**Theorem (Graph Minor Theorem, N. Robertson and P. Seymour 2004).**
Any class of finite graphs closed under minors has a finite FS.

- Proof published in a series of 20 papers spanning 21 years!

# Easy sufficient condition

A preordered set $\langle \mathscr{P}, R \rangle$ is well-quasi-ordered when it is well-founded and contains no infinite antichains.

**Theorem.**

If $\langle \mathscr{P}, R \rangle$ is well-quasi-ordered, then every set closed under $R$ has a finite FS.

- Difficult part of the Graph Minor Theorem: no infinite antichains.

But it is also easy to see that well-quasi-ordered-ness is not necessary.

# Other cardinalities?

Using some heavier-duty Set Theory and Topology, counterexamples to almost all infinite versions of the Graph Minor Theorem have been found:

**Theorem (R. Thomas 1986, P. Komjáth 1995).**

For every $\kappa > \aleph_0$, there exist $2^\kappa$ graphs of size $\kappa$ which form a $\preccurlyeq$-antichain.

- Still an open question for $\kappa = \aleph_0$.

# What about $\leq$ and $\subseteq$?

Changing $\preccurlyeq$ to $\leq$ or $\subseteq$, the "Graph Minor Theorem" is false for all cardinalities.

But given that finite FSs are so nice, we would still like to characterize which sets of graphs closed under those relations have finite FSs.

No such characterization is known yet.

However, if one ever appears, it won't be pretty...

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- Same theorem and proof hold for $\subseteq$.

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Thanks!